Solving Linear Equations

Matrixology (Linear Algebra)—Lecture 2/25
MATH 124, Fall, 2011

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Solving $A \vec{x} = \vec{b}$
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- We (people + computers) solve systems of linear equations by a systematic method of **Elimination** followed by **Back substitution**.
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

  \[-x_1 + 3x_2 = 1 \\
  2x_1 + x_2 = 5 \tag{1}\]

\[\rightsquigarrow \text{chalkage}\]
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Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.

2. Swap rows if needed to create an ‘upper triangular form’
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E.g.

\[ \begin{align*}
2x_1 - x_2 &= -1 \\
x_2 &= 3
\end{align*} \rightarrow \begin{align*}
2x_1 - x_2 &= -1 \\
x_2 &= 3
\end{align*} \]
Solve:

\[ 2x_1 - 3x_2 = 3 \]
\[ 4x_1 - 5x_2 + x_3 = 7 \]
\[ 2x_1 - x_2 - 3x_3 = 5 \]
Summary:
Using **row operations**, we turned this problem:

\[
A\vec{x} = \vec{b} : \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}
\]

into this problem:

\[
U\vec{x} = \vec{d} : \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}
\]

and the latter is **easy to solve using back substitution**.
Defn:
The entries along *U*'s main diagonal are the **pivots** of *A*.
(The pivots are hidden—elimination finds them.)

Defn:
A matrix with only zeros below the main diagonal is called **upper triangular**. A matrix with only zeros above the main diagonal is called **lower triangular**. We travel from *A* to *U* and the latter is always upper triangular.

Defn:
**Singular** means a system has no unique solution.
- It may have no solutions or infinitely many solutions.
- Singular = archaic way of saying ‘messed up.’

Truth:
If at least one pivot is zero, the matrix will be **singular**.
(but the reverse is not necessarily true).
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Gaussian elimination:

**The one true method:**

- We simplify $A$ using elimination in **the same way every time**.
- Eliminate entries one column at a time, moving left to right, and down each column.

\[
\begin{align*}
X + X + X + X &= X \\
1 &\downarrow + X + X + X &= X \\
2 &\downarrow + 4 &\downarrow + X + X &= X \\
3 &\uparrow + 5 &\rightarrow + 6 + X &= X
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Gaussian elimination:

- To eliminate entry in row $i$ of $j$th column, subtract a multiple $\ell_{ij}$ of the $j$th row from $i$.

- For example:

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\begin{align*}
2x_1 & + 3x_2 + -2x_3 + x_4 = 1 \\
x_1 & - 7x_2 + 3x_3 + x_4 = 1 \\
-x_1 & - 3x_2 - x_3 + 5x_4 = -2 \\
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$\ell_{21} = 1/2$, $\ell_{31} = -1/2$, $\ell_{41} = ?$.

- Note: we cannot find $\ell_{32}$ etc., until we are finished with row 1. Pivots are hidden!

- Note: the denominator of each $\ell_{ij}$ multiplier is the pivot in the $j$th column.
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