Introduction
Matrixology (Linear Algebra)—Lecture 1/25
MATH 124, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

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Outline

- Exciting Admin
- Importance
- Usages
- Key problems
- Three ways of looking...
- Colbert on Equations
- References
Basics:

- **Instructor:** Prof. Peter Dodds
- **Lecture room and meeting times:**
  254 Votey Hall,
  Tuesday and Thursday, 2:30 pm to 3:45 pm
- **Office:** Farrell Hall, second floor, Trinity Campus
- **E-mail:** peter.dodds@uvm.edu
- **Course website:** [http://www.uvm.edu/~pdodds/teaching/courses/2011-08UVM-124](http://www.uvm.edu/~pdodds/teaching/courses/2011-08UVM-124)
- **Textbook:** “Introduction to Linear Algebra” (3rd of 4th editions) by Gilbert Strang (published by Wellesley-Cambridge Press).
Our Textbook of Excellence:

- 4th Edition ✓
- 3rd Edition ✓
- Unhelpful □

- “Introduction to Linear Algebra”
  by Gil Strang (.URI);

- Textbook website:
  http://math.mit.edu/linearalgebra/ (URI)

- MIT Open Courseware site for 18.06 (=Linear Algebra):
  http://ocw.mit.edu/...linear-algebra-spring-2010/ (URI)
Yesness:

Money quote from George Cobb’s review of Strang’s book:
Do you want a book written by a mathematician with a lifetime experience using linear algebra to understand important, authentic, applied problems, a former president of the Society for Industrial and Applied Mathematics,

▶ George Cobb: Robert L. Rooke Professor of Mathematics and Statistics, Mount Holyoke College
▶ Full review here [amazon]
Money quote from George Cobb’s review of Strang’s book:

Do you want a book written by a mathematician with a lifetime experience using linear algebra to understand important, authentic, applied problems, a former president of the Society for Industrial and Applied Mathematics, or do you want a book shaped mainly by the [a]esthetics of pure mathematicians with only a weak, theoretical connection to how linear algebra is used in the natural and social sciences?

George Cobb: Robert L. Rooke Professor of Mathematics and Statistics, Mount Holyoke College

Full review here [amazon]
Gil Strang, Exalted Friend of the Matrix:

- Professor of Mathematics at MIT since 1962.

- Many awards including MAA Haimo Award (มง) for Distinguished College or University Teaching of Mathematics

- Rhodes Scholar.

- Legend.

- More on Laplacian matrices, graphs, and other madnesses here (มง).

- (Strang’s Wikipedia page is here (มง)).
Potential paper products:

1. Outline

Papers to read:


Office hours:

- 12:50 pm to 3:50 pm, Wednesday, Farrell Hall, second floor, Trinity Campus
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1. Outline

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Grading breakdown:

1. **Assignments (40%)**
   - Ten one-week assignments.
   - Lowest assignment score will be dropped.
   - The last assignment cannot be dropped!
   - Each assignment will have a random bonus point question which has nothing to do with linear algebra.

2. **Midterm exams (35%)**
   - Three 75 minutes tests distributed throughout the course, all of equal weighting.

3. **Final exam (24%)**
   - ≤ Three hours of joyful celebration.
   - Monday, December 12, 1:30 pm to 4:15 pm, 254 Votey
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4. **Homework (0%)**—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.

5. **General attendance (1%)**—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline.

Questions are worth 3 points according to the following scale:

- 3 = correct or very nearly so.
- 2 = acceptable but needs some revisions.
- 1 = needs major revisions.
- 0 = way off.
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Questions are worth 3 points according to the following scale:

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- 1 = needs major revisions.
- 0 = way off.
The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

<table>
<thead>
<tr>
<th>Week # (dates)</th>
<th>Tuesday</th>
<th>Thursday</th>
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</thead>
<tbody>
<tr>
<td>1 (8/30, 9/1)</td>
<td>Lecture</td>
<td>Lecture + A1</td>
</tr>
<tr>
<td>2 (9/6, 9/8)</td>
<td>Lecture</td>
<td>Lecture + A2</td>
</tr>
<tr>
<td>3 (9/13, 9/15)</td>
<td>Lecture</td>
<td>Lecture + A3</td>
</tr>
<tr>
<td>4 (9/20, 9/22)</td>
<td>Lecture</td>
<td>Test 1</td>
</tr>
<tr>
<td>5 (9/27, 9/29)</td>
<td>Lecture</td>
<td>Lecture + A4</td>
</tr>
<tr>
<td>6 (10/4, 10/6)</td>
<td>Lecture</td>
<td>Lecture + A5</td>
</tr>
<tr>
<td>7 (10/11, 10/13)</td>
<td>Lecture</td>
<td>Lecture + A6</td>
</tr>
<tr>
<td>8 (10/18, 10/20)</td>
<td>Lecture</td>
<td>Test 2</td>
</tr>
<tr>
<td>9 (10/25, 10/27)</td>
<td>Lecture</td>
<td>Lecture + A7</td>
</tr>
<tr>
<td>10 (11/1, 11/3)</td>
<td>Lecture</td>
<td>Lecture + A8</td>
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<tr>
<td>11 (11/8, 11/10)</td>
<td>Lecture</td>
<td>Lecture + A9</td>
</tr>
<tr>
<td>12 (11/15, 11/17)</td>
<td>Lecture</td>
<td>Test 3</td>
</tr>
<tr>
<td>13 (11/22, 11/24)</td>
<td>Thanksgiving</td>
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<tr>
<td>14 (11/29, 12/1)</td>
<td>Lecture + A10</td>
<td>Lecture</td>
</tr>
<tr>
<td>15 (12/6)</td>
<td>Lecture</td>
<td>—</td>
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</table>
Important dates:

1. Classes run from Monday, August 29 to Wednesday, December 7.
3. Last day to withdraw—Monday, October 31 (Boo).
4. Reading and Exam period—Thursday, December 8 to Friday, December 16.

More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.
More stuff:

**Being good people:**

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).

2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.

3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See [http://www.uvm.edu/cses/](http://www.uvm.edu/cses/) for guidelines.
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Even more stuff:

**Late policy:** Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

**Computing:** Students are encouraged to use Matlab or something similar to check their work.

**Note:** for assignment problems, written details of calculations will be required.
Why are we doing this?

Many things are discrete:
- Information (0’s & 1’s, letters, words)
- People (sociology)
- Networks (the Web, people again, food webs, ...)
- Sounds (musical notes)

Even more:

If real data is continuous, we almost always discretize it (0’s and 1’s)
Why are we doing this?

Big deal: **Linear Algebra is** a body of mathematics that deals with **discrete problems**.

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Linear Algebra is used in many fields to solve problems:

- Engineering
- Computer Science
- Physics
- Economics
- Biology
- Ecology...

Big example:
Google’s Pagerank (▶)

Some truth:

- Linear Algebra is as important as Calculus...
- Calculus ≡ the blue pill...
Why are we doing this?

Linear Algebra is used in many fields to solve problems:

- Engineering
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Big example:
Google’s Pagerank ( смысловую нагрузку )

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**Why are we doing this?**

Linear Algebra is used in many fields to solve problems:

| Engineering | Economics |
| Computer Science | Biology |
| Physics | Ecology ...

**Big example:**

Google's Pagerank (-indent)

**Some truth:**

- Linear Algebra is *as important* as Calculus...
- Calculus ≡ the blue pill...
You are now choosing the red pill:
Calculus is the Serpent’s Mathematics.
The Platypus of Truth:

- Platypuses are masters of Linear Algebra.
Linear Algebra:
- Ghandi
- Buffy Summers
- Maple trees
- Chipmunks
- Elephants
- Yoda
- Hermione
- Frodo
- Indiana Jones
- Apple

Calculus:
- Poisonous spiders and other nasty bitey things
- Voldemort
- Big Bads
- Golem
- George Lucas
- Snakes
- Microsoft
A matrix $A$ transforms a vector $\vec{x}$ into a new vector $\vec{x}'$ through matrix multiplication (whatever that is):

$$\vec{x}' = A \vec{x}$$

We can use matrices to:

- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things in different directions
- Reveal the true ur-dystopian reality.
Matrices as gadgets:

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A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^T
Image approximation (80x60)

\[ A = \sum_{i=1}^{2} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{3} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{4} \sigma_i \hat{u}_i \hat{v}_i^T \]
$$A = \sum_{i=1}^{5} \sigma_i \hat{u}_i \hat{v}_i^T$$
Image approximation (80x60)

$$A = \sum_{i=1}^{6} \sigma_i \hat{u}_i \hat{v}_i^T$$
A = \sum_{i=1}^{7} \sigma_i \hat{u}_i \hat{v}_i^T
Image approximation (80x60)

\[ A = \sum_{i=1}^{8} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{9} \sigma_i \hat{u}_i \hat{v}_i^T \]
A = \sum_{i=1}^{10} \sigma_i \hat{u}_i \hat{v}_i^T
A = \sum_{i=1}^{20} \sigma_i \hat{u}_i \hat{v}_i^T
$A = \sum_{i=1}^{30} \sigma_i \hat{u}_i \hat{v}_i^T$
Image approximation (80x60)

\[ A = \sum_{i=1}^{40} \sigma_i \hat{u}_i \hat{v}_i^T \]
A = \sum_{i=1}^{50} \sigma_i \hat{u}_i \hat{v}_i^T
A = \sum_{i=1}^{60} \sigma_i \hat{u}_i \hat{v}_i^T
Best fit line (least squares):

- Linear algebra does this beautifully;
- Calculus version is clunky.

From “Re-examination of the ‘3/4’ law of metabolism”\cite{1}
Dodds, Rothman, and Weitz,
Best fit line (least squares):

- Linear algebra does this beautifully;
- Calculus version is clunky.
  And evil.

From “Re-examination of the ‘3/4’ law of metabolism”\textsuperscript{[1]}
Dodds, Rothman, and Weitz,
The many delights of Eigenthalings:

Using Linear Algebra we’ll somehow connect:

- Fibonacci Numbers,
- Golden Ratio,
- Spirals,
- Sunflowers, pine cones,
- Harvard Square.

Using Linear Algebra we’ll somehow connect:

- Fibonacci Numbers,
- Golden Ratio,
- Spirals,
- Sunflowers, pine cones,
- Harvard Square.
This is a math course:


▶ It’s all connected. “More later.”
Three key problems of Linear Algebra

1. Given a matrix $A$ and a vector $\vec{b}$, find $\vec{x}$ such that

$$A\vec{x} = \vec{b}.$$ 

2. Eigenvalue problem: Given $A$, find $\lambda$ and $\vec{v}$ such that

$$A\vec{v} = \lambda \vec{v}.$$ 

3. Coupled linear differential equations:

$$\frac{d}{dt} y(t) = A y(t)$$

- Our focus will be largely on #1, partly on #2.
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Our focus will be largely on #1, partly on #2.
Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:
Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:

What is going on here? We have 25 24 lectures to find out...
Our new BFF: \( A\vec{x} = \vec{b} \)

Broadly speaking, \( A\vec{x} = \vec{b} \) translates as follows:

- \( \vec{b} \) represents reality (e.g., music, structure)
- \( A \) contains building blocks (e.g., notes, shapes)
- \( \vec{x} \) specifies how we combine our building blocks to make \( \vec{b} \) (as best we can).

How can we disentangle an orchestra’s sound?

- Radiolab’s amazing piece: A 4-Track Mind

What about pictures, waves, signals, …?
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How can we disentangle an orchestra’s sound?

- Radiolab (ازي)’s amazing piece: A 4-Track Mind (ازي)

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How can we disentangle an orchestra’s sound?

- Radiolab ( libido )’s amazing piece: A 4-Track Mind ( libido )

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- Radiolab (梴)’s amazing piece: A 4-Track Mind (梴)

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How can we disentangle an orchestra’s sound?

- Radiolab (확)'s amazing piece: A 4-Track Mind (확)

What about pictures, waves, signals, ...?
Is this your left nullspace?:
Linear Algebra compliments/putdowns for Thanksgiving dinner:

- Wow, you have such a tiny/huge [delete as applicable] left nullspace!
- See also: The Dunning-Kruger effect. (⫶)
Linear Algebra compliments/putdowns for Thanksgiving dinner:

- Wow, you have such a tiny/huge [delete as applicable] left nullspace!
- See also: The Dunning-Kruger effect. (⊞)
Our friend $A\vec{x} = \vec{b}$

What does knowing $\vec{x}$ give us?

- Compress information
- See how we can alter information (filtering)
- Find a system’s simplest representation
- Find a system’s most important elements
- See how to adjust a system in a principled way
Our friend $A\vec{x} = \vec{b}$

**What does knowing $\vec{x}$ give us?**

If we can represent reality as a superposition (or combination or sum) of simple elements, we can do many things:

- Compress information
- See how we can alter information (filtering)
- Find a system’s simplest representation
- Find a system’s most important elements
- See how to adjust a system in a principled way
Our friend $A\vec{x} = \vec{b}$

What does knowing $\vec{x}$ give us?

If we can represent reality as a superposition (or combination or sum) of simple elements, we can do many things:

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- See how we can alter information (filtering)
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- **Way 1: The Row Picture**
- **Way 2: The Column Picture**
- **Way 3: The Matrix Picture**

Example:

\[-x_1 + x_2 = 1\]
\[2x_1 + x_2 = 4\]

- Call this a 2 by 2 system of equations.
- 2 equations with 2 unknowns.
- Standard method of simultaneous equations: solve above by adding and subtracting multiples of equations to each other.
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**Row Picture—what we are doing:**

- (a) Finding intersection of two lines
- (b) Finding the values of $x_1$ and $x_2$ for which both equations are satisfied (true/happy)
- A splendid and deep connection:
  - (a) Geometry $\iff$ (b) Algebra

**Three possible kinds of solution:**

1. Lines intersect at one point
2. Lines are parallel and disjoint
3. Lines are the same
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Three possible kinds of solution:

1. Lines intersect at one point — One, unique solution
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**Three possible kinds of solution:**

1. Lines intersect at one point — **One, unique solution**
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Three possible kinds of solution:

1. Lines intersect at one point — One, unique solution
2. Lines are parallel and disjoint — No solutions
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Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

- Column vectors are our ‘building blocks’
- **Key idea**: try to ‘reach’ $\vec{b}$ by combining (summing) multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$. 

\[ \begin{align*} \vec{x}_1 \vec{a}_1 + \vec{x}_2 \vec{a}_2 &= \vec{b} \\ \begin{bmatrix} -1/2 \end{bmatrix} \vec{x}_1 + \begin{bmatrix} 1/2 \end{bmatrix} \vec{x}_2 &= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \end{align*} \]
Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

See

$$\begin{align*}
-x_1 + x_2 &= 1 \\
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\end{align*}$$

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\[-x_1 + x_2 = 1\]
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\[x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} .\]

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General problem

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Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- Intuitive.
- Generalizes easily to many dimensions.

Three possible kinds of solution:

1. $\vec{a}_1 \parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions
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(assuming neither $\vec{a}_1$ or $\vec{a}_1$ are $\vec{0}$)
Three ways to understand $A\vec{x} = \vec{b}$:

**Difficulties:**

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
  - No! We can still find the $\vec{x}$ that gets us as close to $\vec{b}$ as possible.
- Method of approximation—very important!
- We may not have the right building blocks but we can do our best.
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Three ways to understand $A\vec{x} = \vec{b}$:

**The Matrix Picture:**

$A\vec{x} = \vec{b}$

A is now an operator:

- A transforms $\vec{x}$ into $\vec{b}$.
- Roughly speaking, $A$ does two things to $\vec{x}$:
  1. Rotation/Flipping
  2. Dilation (stretching/contraction)
Three ways to understand $A \vec{x} = \vec{b}$:

The Matrix Picture:
Now see

\[
\begin{align*}
    x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 4 \end{bmatrix}.
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as

$$A\vec{x} = \vec{b} : \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

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- Decomposition or factorization of matrices.
  - Matrices can often be written as products or sums of simpler matrices
  - $A = LU$, $A = QR$, $A = U\Sigma V^T$, $A = \sum_i \lambda_i \vec{v} \vec{v}^T$, ...

Lecture 1/25: Introduction

Exciting Admin
Importance
Usages
Key problems
Three ways of looking...
Colbert on Equations
References
The Matrix Picture

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Key idea in linear algebra:

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\[ A = LU, \quad A = QR, \quad A = U\Sigma V^T, \quad A = \sum_i \lambda_i \vec{v}_i \vec{v}_i^T, \ldots \]
“Equations are the Devil’s sentences.”
