Introduction
Matrixology (Linear Algebra)—Lecture 1/25
MATH 124, Fall, 2011

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Our Textbook of Excellence:

4th Edition
3rd Edition
Unhelpful

Yesness:
Money quote from George Cobb's review of Strang's book:
Do you want a book written by a mathematician with a lifetime experience using linear algebra to understand important, authentic, applied problems, a former president of the Society for Industrial and Applied Mathematics, or do you want a book shaped mainly by the [a]esthetics of pure mathematicians with only a weak, theoretical connection to how linear algebra is used in the natural and social sciences?

George Cobb: Robert L. Rooke Professor of Mathematics and Statistics, Mount Holyoke College
Full review here [amazon]

Gil Strang, Exalted Friend of the Matrix:
Professor of Mathematics at MIT since 1962.

Basics:
Instructor: Prof. Peter Dodds
Lecture room and meeting times:
254 Votey Hall, Tuesday and Thursday, 2:30 pm to 3:45 pm
Office: Farrell Hall, second floor, Trinity Campus
E-mail: peter.dodds@uvm.edu
Course website: http://www.uvm.edu/~pdodds/teaching/courses/2011-08UVM-124
The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

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Important dates:
1. Classes run from Monday, August 29 to Wednesday, December 7.
3. Last day to withdraw—Monday, October 31 (Boo).
4. Reading and Exam period—Thursday, December 8 to Friday, December 16.

More stuff:
Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.

Being good people:
1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).
2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.
3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See http://www.uvm.edu/cses/ for guidelines.
Even more stuff:

Late policy: Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

Computing: Students are encouraged to use Matlab or something similar to check their work.

Note: for assignment problems, written details of calculations will be required.

Why are we doing this?

Big deal: **Linear Algebra** is a body of mathematics that deals with *discrete problems*.

Many things are discrete:
- Information (0's & 1's, letters, words)
- People (sociology)
- Networks (the Web, people again, food webs, ...)
- Sounds (musical notes)

Even more:

If real data is continuous, we almost always discretize it (0's and 1's)

Why are we doing this?

**Linear Algebra** is used in many fields to solve problems:
- Engineering
- Computer Science
- Physics
- Economics
- Biology
- Ecology ...

Big example: Google's Pagerank (셑)

Some truth:
- **Linear Algebra** is as important as Calculus...
- **Calculus** ≡ the blue pill...

The Truth:

- Calculus is the Serpent's Mathematics.

The Platypus of Truth:

- Platypuses are masters of Linear Algebra.
The Truth:

Linear Algebra:
- Ghandi
- Buffy Summers
- Maple trees
- Chipmunks
- Elephants
- Yoda
- Hermione
- Frodo
- Indiana Jones
- Apple

Calculus:
- Poisonous spiders and other nasty bitey things
- Voldemort
- Big Bads
- Golem
- George Lucas
- Snakes
- Microsoft

Matrices as gadgets:
A matrix $A$ transforms a vector $\vec{x}$ into a new vector $\vec{x}'$ through matrix multiplication (whatever that is):

$$\vec{x}' = A \vec{x}$$

We can use matrices to:
- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things in different directions
- Reveal the true dystopian reality.

Best fit line (least squares):

- Linear algebra does this beautifully;
- Calculus version is clunky.
- And evil.


The many delights of Eigenthings:

Using Linear Algebra we’ll somehow connect:

- Fibonacci Numbers,
- Golden Ratio,
- Spirals,
- Sunflowers, pine cones, ...
- Harvard Square.

This is a math course:

- It’s all connected. “More later.”

Three key problems of Linear Algebra

1. Given a matrix $A$ and a vector $\vec{b}$, find $\vec{x}$ such that $A \vec{x} = \vec{b}$.

2. Eigenvalue problem: Given $A$, find $\lambda$ and $\vec{v}$ such that $A \vec{v} = \lambda \vec{v}$.

3. Coupled linear differential equations:

$$\frac{d}{dt} y(t) = A y(t)$$

- Our focus will be largely on #1, partly on #2.

References

[1] [Link to the paper]
Major course objective:
To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:

What is going on here? We have 25-24 lectures to find out...

Our new BFF: $A\vec{x} = \vec{b}$

Broadly speaking, $A\vec{x} = \vec{b}$ translates as follows:
- $\vec{b}$ represents reality (e.g., music, structure)
- $A$ contains building blocks (e.g., notes, shapes)
- $\vec{x}$ specifies how we combine our building blocks to make $\vec{b}$ (as best we can).

How can we disentangle an orchestra’s sound?
- Radiolab (⊞)'s amazing piece: A 4-Track Mind (⊞)

What about pictures, waves, signals, ...?

Is this your left nullspace?:

Our friend $A\vec{x} = \vec{b}$

What does knowing $\vec{x}$ give us?
If we can represent reality as a superposition (or combination or sum) of simple elements, we can do many things:
- Compress information
- See how we can alter information (filtering)
- Find a system’s simplest representation
- Find a system’s most important elements
- See how to adjust a system in a principled way

Three ways to understand $A\vec{x} = \vec{b}$:

- Way 1: The Row Picture
- Way 2: The Column Picture
- Way 3: The Matrix Picture

Example:

- $-x_1 + x_2 = 1$
- $2x_1 + x_2 = 4$

- Call this a 2 by 2 system of equations.
- 2 equations with 2 unknowns.
- Standard method of simultaneous equations: solve above by adding and subtracting multiples of equations to each other = Row Picture.
Three ways to understand $A\vec{x} = \vec{b}$:

Row Picture—what we are doing:

- (a) Finding intersection of two lines
- (b) Finding the values of $x_1$ and $x_2$ for which both equations are satisfied (true/happy)
- A splendid and deep connection: (a) Geometry $\iff$ (b) Algebra

Three possible kinds of solution:
1. Lines intersect at one point — One, unique solution
2. Lines are parallel and disjoint — No solutions
3. Lines are the same — Infinitely many solutions

Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

See

$$-x_1 + x_2 = 1$$
$$2x_1 + x_2 = 4$$

as

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

General problem

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

- Column vectors are our ‘building blocks’
- Key idea: try to ‘reach’ $\vec{b}$ by combining (summing) multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$.

Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- Intuitive.
- Generalizes easily to many dimensions.

Three possible kinds of solution:
1. $\vec{a}_1 \parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions

(assuming neither $\vec{a}_1$ or $\vec{a}_2$ are $\vec{0}$)

Three ways to understand $A\vec{x} = \vec{b}$:

The Matrix Picture:

Now see

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

as

$$A\vec{x} = \vec{b}: \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

$A$ is now an operator:

- $A$ transforms $\vec{x}$ into $\vec{b}$.
- Roughly speaking, $A$ does two things to $\vec{x}$:
  1. Rotation/Flipping
  2. Dilation (stretching/contraction)

Key idea in linear algebra:

- Decomposition or factorization of matrices.
- Matrices can often be written as products or sums of simpler matrices
  - $A = LU$, $A = QR$, $A = UΣV^T$, $A = \sum \lambda_i \vec{v}_i \vec{v}_i^T$, ...

Difficulties:

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
- No! We can still find the $\vec{x}$ that gets us as close to $\vec{b}$ as possible.
- Method of approximation—very important!
- We may not have the right building blocks but we can do our best.

References

Equations

Colbert on

References

Equations

Colbert on

References

Equations

Colbert on

References

Equations
“Equations are the Devil’s sentences.”

References

