Outline

Random walks on networks

Random walks on networks—basics:
- Imagine a single random walker moving around on a network.
- At \( t = 0 \), start walker at node \( j \) and take time to be discrete.
- \( Q \): What’s the long term probability distribution for where the walker will be?
- Define \( \rho_i(t) \) as the probability that at time step \( t \), our walker is at node \( i \).
- We want to characterize the evolution of \( \dot{\rho}(t) \).
- First task: connect \( \dot{\rho}(t + 1) \) to \( \dot{\rho}(t) \).
- Let’s call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is hopelessly drunk.

Where is Barry?
- Consider simple undirected, ergodic (strongly connected) networks.
- As usual, represent network by adjacency matrix \( A \) where
  \[
  a_{ij} = 1 \text{ if } i \text{ has an edge leading to } j, \\
  a_{ij} = 0 \text{ otherwise.}
  \]
- Barry is at node \( j \) at time \( t \) with probability \( p_j(t) \).
- In the next time step, he randomly lurches toward one of \( j \)'s neighbors.
- Barry arrives at node \( i \) from node \( j \) with probability \( \frac{1}{k_j} \) if an edge connects \( j \) to \( i \).
- Equation-wise:
  \[
  p_i(t + 1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} p_j(t).
  \]
  where \( k_j \) is \( j \)'s degree. Note: \( k_i = \sum_{j=1}^{n} a_{ij} \).

Inebriation and diffusion:
- \( x_i(t) = \) amount of stuff at node \( i \) at time \( t \).
- \[
  x_i(t + 1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} x_j(t).
  \]
- Random walking is equivalent to diffusion (\( \Box \)).

Where is Barry?
- Linear algebra-based excitement:
  \[
  \rho_i(t + 1) = \sum_{j=1}^{n} a_{ij} \frac{1}{k_j} \rho_j(t)
  \]
  is more usefully viewed as
  \[
  \dot{\rho}(t + 1) = A^T K^{-1} \dot{\rho}(t)
  \]
  where \( [K] = [k_j] \) has node degrees on the main diagonal and zeros everywhere else.
- So... we need to find the dominant eigenvalue of \( A^T K^{-1} \).
- Expect this eigenvalue will be 1 (doesn’t make sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.
Where is Barry?

- By inspection, we see that

\[ \vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{K} \]

satisfies \[ \vec{p}(\infty) = A^T K^{-1} \vec{p}(\infty) \] with eigenvalue 1.

- We will find Barry at node \( i \) with probability proportional to its degree \( k_i \).

- Nice implication: probability of finding Barry travelling along any edge is uniform.

- Diffusion in real space smooths things out.

- On networks, uniformity occurs on edges.

- So in fact, diffusion in real space is about the edges too but we just don’t see that.

Other pieces:

- Goodness: \( A^T K^{-1} \) is similar to a real symmetric matrix if \( A = A^T \).

- Consider the transformation \( M = K^{-1/2} \):

\[ K^{-1/2} A^T K^{-1} K^{1/2} = K^{-1/2} A^T K^{-1/2} \]

- Since \( A^T = A \), we have

\[ (K^{-1/2} AK^{-1/2})^T = K^{-1/2} AK^{-1/2} \]

- Upshot: \( A^T K^{-1} = AK^{-1} \) has real eigenvalues and a complete set of orthogonal eigenvectors.

- Can also show that maximum eigenvalue magnitude is indeed 1.