Random walks and diffusion on networks

Complex Networks
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Outline

Random walks on networks
Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node $j$ and take time to be discrete.
- Q: What’s the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\bar{p}(t)$.
- First task: connect $\bar{p}(t + 1)$ to $\bar{p}(t)$.
- Let’s call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is hopelessly drunk.
Where is Barry?

- Consider simple undirected, ergodic (strongly connected) networks.
- As usual, represent network by adjacency matrix $A$ where
  \[
  a_{ij} = 1 \text{ if } i \text{ has an edge leading to } j, \\
  a_{ij} = 0 \text{ otherwise.}
  \]
- Barry is at node $j$ at time $t$ with probability $p_j(t)$.
- In the next time step, he randomly lurches toward one of $j$’s neighbors.
- Barry arrives at node $i$ from node $j$ with probability $\frac{1}{k_j}$ if an edge connects $j$ to $i$.
- Equation-wise:
  \[
  p_i(t + 1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} p_j(t).
  \]
  where $k_j$ is $j$’s degree. Note: $k_i = \sum_{j=1}^{n} a_{ij}$.
Inebriation and diffusion:

- **Excellent observation**: The same equation applies for stuff moving around a network, such that at each time step all material at node $i$ is sent to its neighbors.

  $x_i(t) = \text{amount of stuff at node } i \text{ at time } t.$

- Random walking is equivalent to diffusion ($\triangleright$)
Where is Barry?

- Linear algebra-based excitement:
  \[ p_i(t + 1) = \sum_{j=1}^{n} a_{ji} \frac{1}{k_j} p_j(t) \]
  is more usefully viewed as
  \[ \vec{p}(t + 1) = A^T K^{-1} \vec{p}(t) \]
  where \([K_{ij}] = [\delta_{ij} k_i]\) has node degrees on the main diagonal and zeros everywhere else.

- So... we need to find the **dominant eigenvalue** of \(A^T K^{-1}\).

- Expect this eigenvalue will be 1 (doesn’t make sense for total probability to change).

- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).

- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.
Where is Barry?

- By inspection, we see that
  \[
  \vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k}
  \]
  satisfies \( \vec{p}(\infty) = A^T K^{-1} \vec{p}(\infty) \) with eigenvalue 1.

- We will find Barry at node \( i \) with probability proportional to its degree \( k_i \).

- Nice implication: probability of finding Barry travelling along any edge is uniform.

- Diffusion in real space smooths things out.

- On networks, uniformity occurs on edges.

- So in fact, diffusion in real space is about the edges too but we just don’t see that.
Other pieces:

- Goodness: $A^T K^{-1}$ is similar to a real symmetric matrix if $A = A^T$.
- Consider the transformation $M = K^{-1/2}$:
  \[
  K^{-1/2} A^T K^{-1} K^{1/2} = K^{-1/2} A^T K^{-1/2}.
  \]
- Since $A^T = A$, we have
  \[
  (K^{-1/2} AK^{-1/2})^T = K^{-1/2} AK^{-1/2}.
  \]
- Upshot: $A^T K^{-1} = AK^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.