Measures of centrality

Complex Networks
CSYS/MATH 303, Spring, 2011

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References
Outline

Background

Centrality measures
Degree centrality
Closeness centrality
Betweenness
Eigenvalue centrality
Hubs and Authorities

References
How big is my node?

- **Basic question:** how ‘important’ are specific nodes and edges in a network?

- An important node or edge might:
  1. handle a relatively large amount of the network’s traffic (e.g., cars, information);
  2. bridge two or more distinct groups (e.g., liason, interpreter);
  3. be a source of important ideas, knowledge, or judgments (e.g., supreme court decisions, an employee who ‘knows where everything is’).

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- We generate ad hoc, reasonable measures, and examine their utility...
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One possible reflection of importance is **centrality**.

Presumption is that nodes or edges that are (in some sense) in the middle of a network are important for the network’s function.

Idea of centrality comes from social networks literature [7].

Many flavors of centrality...

1. Many are topological and quasi-dynamical;
2. Some are based on dynamics (e.g., traffic).

We will define and examine a few...

(Later: see centrality useful in identifying communities in networks.)
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- Naively estimate importance by node degree.\textsuperscript{[7]}
- Doh: assumes linearity
  (If node $i$ has twice as many friends as node $j$, it’s twice as important.)
- Doh: doesn’t take in any non-local information.
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- **Idea:** Nodes are more central if they can reach other nodes ‘easily.’
- Measure average shortest path from a node to all other nodes.
- Define Closeness Centrality for node $i$ as
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  \frac{N - 1}{\sum_{j \neq i} \text{(distance from } i \text{ to } j)}.\]
- Range is 0 (no friends) to 1 (single hub).
- Unclear what the exact values of this measure tells us because of its ad-hocness.
- General problem with simple centrality measures: what do they exactly mean?
- Perhaps, at least, we obtain an ordering of nodes in terms of ‘importance.’
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  - **Idea:** If the quickest way between any two nodes on a network disproportionately involves certain nodes, then they are ‘important’ in terms of global cohesion.
  
  - For each node $i$, count how many shortest paths pass through $i$.
  
  - In the case of ties, or divide counts between paths.
  
  - Call frequency of shortest paths passing through node $i$ the betweenness of $i$, $B_i$.
  
  - **Note:** Exclude shortest paths between $i$ and other nodes.
  
  - **Note:** Works for weighted and unweighted networks.
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Consider a network with $N$ nodes and $m$ edges (possibly weighted).

Computational goal: Find $\binom{N}{2}$ shortest paths between all pairs of nodes.

Traditionally use Floyd-Warshall algorithm.

Computation time grows as $O(N^3)$.

See also:

1. Dijkstra’s algorithm for finding shortest path between two specific nodes,
2. and Johnson’s algorithm which outperforms Floyd-Warshall for sparse networks: $O(mN + N^2 \log N)$.

Newman (2001) and Brandes (2001) independently derive equally fast algorithms that also compute betweenness.

Computation times grow as:

1. $O(mN)$ for unweighted graphs;
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Shortest path between node $i$ and all others:

- Consider unweighted networks.
- Use breadth-first search:
  1. Start at node $i$, giving it a distance $d = 0$ from itself.
  2. Create a list of all of $i$’s neighbors and label them being at a distance $d = 1$.
  3. Go through list of most recently visited nodes and find all of their neighbors.
  4. Exclude any nodes already assigned a distance.
  5. Increment distance $d$ by 1.
  6. Label newly reached nodes as being at distance $d$.
  7. Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from $i$ (former are ‘predecessors’ with respect to $i$’s shortest path structure).
- Runs in $O(m)$ time and gives $N - 1$ shortest paths.
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  1. $j$ indexes edges,
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(a) and (b) illustrate the calculation of shortest-path betweenness. In (a), the vertex $s$ is the source. The leaves are labeled with 1, and the other vertices are labeled with the sum of the scores on the neighboring edges. In (b), the leaves are labeled with $\frac{11}{6}$ and $\frac{25}{6}$, respectively.

The Newman’s Betweenness algorithm:

1. **A. Shortest-path betweenness**
   - Calculating the shortest path between a particular pair of vertices can be done using breadth-first search in time $O(nm)$, where $n$ is the number of vertices and $m$ is the number of edges. Finding all betweennesses in a graph can be done using a variation of this approach.
   - For the vertex $s$, the leaves are labeled with 1, and the other vertices are labeled with the sum of the scores on the neighboring edges.
   - In the simplest case, when there is only a single shortest path from the source vertex, the vertex $s$ is the source, and the leaves are labeled with 1.

2. **B. Finding and removing the highest-scoring edge**
   - Finding and removing the highest-scoring edge is trivial and does not computationally demanding.

3. **C. Recalculation step**
   - The recalculation step is absolutely crucial to the operation of the cases illustrated here.

4. **D. Newman’s Betweenness algorithm**
   - The Newman’s Betweenness algorithm is based on the shortest-path method. It is the only version of the algorithm they should use for their own problem, let us give an immediate answer: for most problems, we recommend the algorithm of Newman.
   - The Newman’s Betweenness algorithm is as follows:
     - For each vertex $v$, we calculate the number of shortest paths that pass through $v$.
     - We then sum these numbers for all vertices.
   - The Newman’s Betweenness algorithm is described in Sec. III A, it can be calculated for all edges in $O(n^2)$ time.

5. **E. Comparison with other algorithms**
   - New algorithms have been proposed by Newman and Girvan, and Freeman vertex betweenness, but it is trivial to adapt their algorithms for edge betweenness. We describe the standard Freeman vertex betweenness, but it is trivial to calculate faster than this, finding all betweennesses in time $O(mn^2)$.

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- Assume further that constant of proportionality, $c$, is independent of $i$.
- Above gives $\vec{x} = cA^T\vec{x}$ or $A^T\vec{x} = c^{-1}\vec{x} = \lambda \vec{x}$.

- Eigenvalue equation based on adjacency matrix...
- **Note:** Lots of despair over size of the largest eigenvalue. \[7\] Lose sight of original assumption’s non-physicality.
Important nodes have important friends:

- Define $x_i$ as the ‘importance’ of node $i$.
- **Idea**: $x_i$ depends (somehow) on $x_j$ if $j$ is a neighbor of $i$.
- **Recursive**: importance is transmitted through a network.
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- So... solve $A^T \vec{x} = \lambda \vec{x}$.
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- We, the people, would like:
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  6. Values of $x_i$ to mean something
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Perron-Frobenius theorem: (⑦)

If an $N \times N$ matrix $A$ has non-negative entries then:

1. $A$ has a real eigenvalue $\lambda_1 \geq |\lambda_i|$ for $i = 2, \ldots, N$.
2. $\lambda_1$ corresponds to left and right 1-d eigenspaces for which we can choose a basis vector that has non-negative entries.
3. The dominant real eigenvalue $\lambda_1$ is bounded by the minimum and maximum row sums of $A$:

   $$\min_j \sum_{i=1}^{N} a_{ij} \leq \lambda_1 \leq \max_i \sum_{j=1}^{N} a_{ij}$$

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- Assuming our network is irreducible, meaning there is only one component, is reasonable: just consider one component at a time if more than one exists.
- Irreducibility means largest eigenvalue's eigenvector has strictly non-negative entries.
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Outline

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Betweenness
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Generalize eigenvalue centrality to allow nodes to have two attributes:

1. **Authority**: how much knowledge, information, etc., held by a node on a topic.
2. **Hubness** (or **Hubosity** or **Hubbishness**): how well a node ‘knows’ where to find information on a given topic.

- Original work due to the legendary Jon Kleinberg.\(^2\)
- Best hubs point to best authorities.
- **Recursive**: nodes can be both hubs and authorities.
- **More**: look for dense links between sets of good hubs pointing to sets of good authorities.
- Known as the **HITS algorithm** (Hyperlink-Induced Topics Search).
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- As for eigenvector centrality, we connect the scores of neighboring nodes.

- New story I: a good authority is linked to by good hubs.
- Means \( x_i \) should increase as \( \sum_{j=1}^{N} a_{ji} y_j \) increases.
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  1. \( x_i \) = authority score for node \( i \)
  2. \( y_i \) = hubtasticness score for node \( i \)

- As for eigenvector centrality, we connect the scores of neighboring nodes.

- New story I: a good authority is linked to by good hubs.
  - Means \( x_i \) should increase as \( \sum_{j=1}^{N} a_{ji} y_j \) increases.
  - Note: indices are \( ji \) meaning \( j \) has a directed link to \( i \).

- New story II: good hubs point to good authorities.
  - Means \( y_i \) should increase as \( \sum_{j=1}^{N} a_{ij} x_j \) increases.

- Linearity assumption:
  \[ \vec{x} \propto A^T \vec{y} \text{ and } \vec{y} \propto A \vec{x} \]
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So let’s say we have
\[ \vec{x} = c_1 A^T \vec{y} \] and \[ \vec{y} = c_2 A \vec{x} \]
where \( c_1 \) and \( c_2 \) must be positive.

Above equations combine to give
\[ \vec{x} = c_1 A^T c_2 A \vec{x} = \lambda A^T A \vec{x} \]
where \( \lambda = c_1 c_2 > 0 \).

It’s all good: we have the heart of singular value decomposition before us...
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We can do this:

- $A^T A$ is symmetric.
- $A^T A$ is semi-positive definite so its eigenvalues are all $\geq 0$.
- $A^T A$’s eigenvalues are the square of $A$’s singular values.
- $A^T A$’s eigenvectors form a joyful orthogonal basis.
- Perron-Frobenius tells us that only the dominant eigenvalue’s eigenvector can be chosen to have non-negative entries.
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References I

A faster algorithm for betweenness centrality.  

Authoritative sources in a hyperlinked environment.  
*Proc. 9th ACM-SIAM Symposium on Discrete Algorithms*, 1998. pdf (isEnabled)

An elementary proof of the perron-frobenius theorem for non-negative symmetric matrices.  
References II

Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality.

Finding and evaluating community structure in networks.

A simple proof of the Perron-Frobenius theorem for positive symmetric matrices.

Social Network Analysis: Methods and Applications.