Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- In terms of network architecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's laws have four parameters and Tokunaga has two parameters.
- \( R_m, R_n, R_i, \) and \( R_s \) versus \( T_1 \) and \( R_T \). One simple redundancy: \( R_i = R_s \). Insert question 2, assignment 2 (**) 
- To make a connection, clearest approach is to start with Tokunaga's law...
- Known result: Tokunaga \( \rightarrow \) Horton [18, 19, 20, 9, 2]

Let us make them happy

We need one more ingredient:

Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks: Drainage density \( \rho_{dd} = \text{inverse of typical distance between channels in a landscape.} \)
- In terms of basin characteristics:

\[
\rho_{dd} \approx \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} \bar{n}_{\omega} s_{\omega}}{a_\Omega}
\]

More with the happy-making thing

Start with Tokunaga's law: \( T_k = T_1 R_T^{k-1} \)

- Start looking for Horton's stream number law:

\[
\frac{n_{\omega}}{n_{\omega+1}} = R_m^n
\]

- Estimate \( n_{\omega} \), the number of streams of order \( \omega \) in terms of other \( n_{\omega'}, \omega' > \omega \).
- Observe that each stream of order \( \omega \) terminates by either:

1. Running into another stream of order \( \omega \) and generating a stream of order \( \omega + 1 \)... 
   - \( 2n_{\omega+1} \) streams of order \( \omega \) do this
2. Running into and being absorbed by a stream of higher order \( \omega' > \omega \)... 
   - \( n_{\omega'} T_{\omega' - \omega} \) streams of order \( \omega \) do this

More with the happy-making thing

Putting things together:

- \( n_\omega = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_{\omega' - \omega} n_{\omega'} \) generation absorption

- Use Tokunaga's law and manipulate expression to create \( R_m \)’s.
- Insert question 3, assignment 2 (**)
- Solution:

\[
R_m = \left( \frac{2 + R_T + T_1}{2} \right) \pm \sqrt{\left(2 + R_T + T_1\right)^2 - 8R_T}
\]

(The larger value is the one we want.)
**Finding other Horton ratios**

Connect Tokunaga to \( R_s \)
- Now use uniform drainage density \( \rho_{dd} \).
- Assume side streams are roughly separated by distance \( 1/\rho_{dd} \).
- For an order \( \omega \) stream segment, expected length is
  \[
  \bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + \sum_{k=1}^{\omega-1} T_k \right)
  \]
- Substitute in Tokunaga's law \( T_n = T_1 R_T^{k-1} \):
  \[
  \bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^\omega
  \]

Horton and Tokunaga are happy

Altogether then:
- \( \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T \)

Recall \( R_1 = R_s \) so

\[
R_s = R_s = R_T
\]

And from before:

\[
R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8 R_T}}{2}
\]

Horton and Tokunaga are friends

Some observations:
- \( R_n \) and \( R_1 \) depend on \( T_1 \) and \( R_T \).
- Seems that \( R_s \) must as well...
- Suggests Horton's laws must contain some redundancy
- We'll in fact see that \( R_s = R_n \).
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions.\(^3\)\(^4\)

Horton and Tokunaga are happy

The other way round
- Note: We can invert the expressions for \( R_s \) and \( R_1 \) to find Tokunaga's parameters.
  \[
  R_T = R_s,
  \]
  \[
  T_1 = R_n - R_1 - 2 + 2 R_s / R_n.
  \]
- Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform)...
Horton and Tokunaga are friends

Just checking:

- Substitute Tokunaga’s law $T_i = T_1 R_i^{\ell - 1} = T_1 R_i^{k - 1}$ into

$$T_k = (R_k - 1) \left( 1 + \sum_{i=1}^{k-1} T_i \right)$$

- $T_1 = (R_1 - 1) \left( 1 + T_1 R_1^{k-1} - 1 \right)$

$\simeq (R_k - 1) T_1 R_1^{k-1} R_k^{k-1} = T_1 R_1^{k-1}$ ... yep.

Horton’s laws of area and number:

- In right plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_d$...

Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

### Mississippi:

<table>
<thead>
<tr>
<th>$\omega$ range</th>
<th>$R_n$</th>
<th>$R_d$</th>
<th>$R_i$</th>
<th>$R_s$</th>
<th>$R_{sd}/R_n$</th>
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<td>2.41</td>
<td>2.56</td>
<td>1.12</td>
</tr>
</tbody>
</table>

- Mean $\mu$: 4.69, 4.85, 2.40, 2.33, 1.04
- Std dev $\sigma$: 0.21, 0.13, 0.04, 0.07, 0.03

### Amazon:

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<tr>
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</tr>
</tbody>
</table>

- Mean $\mu$: 4.42, 4.53, 2.25, 2.10, 1.02
- Std dev $\sigma$: 0.17, 0.10, 0.10, 0.09, 0.02

Reducing Horton’s laws:

Rough first effort to show $R_n \equiv R_d$:

- $a_\Omega \propto$ sum of all stream segment lengths in an order $\Omega$ basin (assuming uniform drainage density)

So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega / \rho a$$

$$\propto \sum_{\omega=1}^{\Omega} R_\omega^{\Omega-\omega} R_{\omega-1} = R_\omega^{\Omega-\omega} R_{\omega-1}$$

$$= \frac{R_\omega^{\Omega-\omega}}{R_s} R_{\omega-1} \sum_{\omega=1}^{\Omega} (R_s / R_n)^\omega$$
Reducing Horton’s laws:

Continued ...

\[ a_\omega \propto \frac{R_{\Omega}^2}{R_a} \bar{s}_1 \sum_{\omega \geq 1} \left( \frac{R_s}{R_a} \right)^\omega \]

\[ = \frac{R_{\Omega}^2}{R_a} \bar{s}_1 \left( \frac{R_s}{R_a} \right) \frac{1}{1 - \left( \frac{R_s}{R_a} \right)^\omega} \]

\[ \sim R_{\Omega}^{\omega-1} \bar{s}_1 \frac{1}{1 - \left( \frac{R_s}{R_a} \right)} \text{ as } \Omega \]

\[ \Rightarrow a_\omega \text{ is growing like } R_{\Omega}^\omega \text{ and therefore: } R_{\Omega} \equiv R_a \]

---

Equipartitioning:

Intriguing division of area:

- Observe: Combined area of basins of order \( \omega \) independent of \( \omega \).
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.

Story:

\[ R_{\Omega} \equiv R_a \Rightarrow n_\omega \bar{a}_\omega = \text{const} \]

Reason:

\[ n_\omega \propto (R_a)^{\omega} \]

\[ \bar{a}_\omega \propto (R_s)^{\omega} \propto n_\omega^{-1} \]

---

Scaling laws

A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network \( p \).
- Each point \( p \) is associated with a basin and a longest stream length

Q: What is probability that the \( p \)'s drainage basin has area \( a \)?

\[ P(a) \propto a^{-\gamma} \text{ for large } a \]

Q: What is probability that the longest stream from \( p \) has length \( \ell \)?

\[ P(\ell) \propto \ell^{-\gamma} \text{ for large } \ell \]

Roughly observed: \( 1.3 \leq \gamma \leq 1.5 \) and \( 1.7 \leq \gamma \leq 2.0 \)
Scaling laws

Probability distributions with power-law decays
► We see them everywhere:
  ► Earthquake magnitudes (Gutenberg-Richter law)
  ► City sizes (Zipf’s law)
  ► Word frequency (Zipf’s law)\(^{[21]}\)
  ► Wealth (maybe not—at least heavy tailed)
  ► Statistical mechanics (phase transitions)\(^{[5]}\)
► A big part of the story of complex systems
► Arise from mechanisms: growth, randomness, optimization, ...
► Our task is always to illuminate the mechanism...

Finding γ:
► The connection between \(P(x)\) and \(P_\geq(x)\) when \(P(x)\) has a power law tail is simple:

\[
P_\geq(\ell_\ast) = \int_{\ell=\ell_\ast}^{\ell_{\text{max}}} P(\ell) \, d\ell
\]

\[
\sim \int_{\ell=\ell_\ast}^{\ell_{\text{max}}} \ell^{-\gamma} \, d\ell
\]

\[
= \frac{\ell^{-\gamma+1}}{-\gamma+1} \int_{\ell=\ell_\ast}^{\ell_{\text{max}}} \ell \, d\ell
\]

\[
\propto \ell^{-\gamma+1} \text{ for } \ell_{\text{max}} \gg \ell_\ast
\]

Scaling laws

Connecting exponents
► We have the detailed picture of branching networks (Tokunaga and Horton)
► Plan: Derive \(P(a) \propto a^{-\gamma}\) and \(P(\ell) \propto \ell^{-\gamma}\) starting with Tokunaga/Horton story\(^{[17, 1, 2]}\)
► Let’s work on \(P(\ell)\)...
► Our first fudge: assume Horton’s laws hold throughout a basin of order \(\Omega\).
  (We know they deviate from strict laws for low \(\omega\) and high \(\omega\) but not too much.)
► Next: place stick between teeth. Bite stick. Proceed.

Finding γ:
► Aim: determine probability of randomly choosing a point on a network with main stream length \(> \ell_\ast\)
► Assume some spatial sampling resolution \(\Delta\)
► Landscape is broken up into grid of \(\Delta \times \Delta\) sites
► Approximate \(P_\geq(\ell_\ast)\) as

\[
P_\geq(\ell_\ast) = \frac{N_\ast(\ell_\ast; \Delta)}{N_\ast(0; \Delta)}
\]

where \(N_\ast(\ell_\ast; \Delta)\) is the number of sites with main stream length \(> \ell_\ast\).
► Use Horton’s law of stream segments:

\[
\frac{n_\omega}{n_{\omega-1}} = R_\ell...
\]

Scaling laws

Finding γ:
► Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
► The complementary cumulative distribution turns out to be most useful:

\[
P_\geq(\ell_\ast) = P(\ell > \ell_\ast) = \int_{\ell=\ell_\ast}^{\ell_{\text{max}}} P(\ell) \, d\ell
\]

\[
P_\geq(\ell_\ast) = 1 - P(\ell < \ell_\ast)
\]

Also known as the exceedance probability.

Finding γ:
► Set \(\ell_\ast = \ell_\omega\) for some \(1 \ll \omega \ll \Omega\).

\[
P_\geq(\ell_\omega) = \frac{N_\ast(\ell_\omega; \Delta)}{N_\ast(0; \Delta)} = \frac{\sum_{\omega'\omega=\omega+1} n_{\omega'} s_{\omega'}}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}}
\]

\(\Delta\)’s cancel
► Denominator is a \(\text{const} \times a_{100}\Delta\), a constant.
► So... using Horton’s laws...

\[
P_\geq(\ell_\omega) \propto \sum_{\omega'=\omega+1} n_{\omega'} s_{\omega'} = \sum_{\omega'=\omega+1} \left(1 - R_0^{\omega'-\omega}\right)(\Omega_1 R_0^{-\omega'-1})
\]
Scaling laws

Finding γ:

- We are here:
  \[ P_x(\ell) \propto \sum_{\omega=1}^{\Omega} (1 \cdot R_n^{\Omega-\omega})(\bar{s}_1 \cdot R_s^{\omega-1}) \]
- Cleaning up irrelevant constants:
  \[ P_x(\ell) \propto \sum_{\omega=\omega+1}^{\Omega} \left( \frac{R_s}{R_n} \right)^{\omega} \]
- Change summation order by substituting
  \[ \omega'' = \Omega - \omega' \]
- Sum is now from \( \omega'' = 0 \) to \( \omega'' = \Omega - \omega - 1 \) (equivalent to \( \omega' = \Omega \) down to \( \omega' = \omega + 1 \))

Scaling laws

Finding γ:

- Since \( R_n > R_s \) and \( 1 < \omega \ll \Omega \),
  \[ P_x(\ell) \propto \left( \frac{R_n}{R_s} \right)^{\Omega-\omega} \left( \frac{R_n}{R_s} \right)^{-\omega} \]
  again using \( \sum_{\omega=0}^{\Omega-1} a' = (a^n - 1)/(a - 1) \)

Hack's law:[6]

- Typically observed that \( 0.5 \lesssim h \lesssim 0.7 \).
  - Use Horton laws to connect \( h \) to Horton ratios:
    \[ \ell_\omega \propto R_s^{\omega} \quad \text{and} \quad a_\omega \propto R_n^{\omega} \]
  - Observe:
    \[ \ell_\omega \propto e^{\omega \ln R_n} \propto \left( \frac{R_n^{\omega}}{R_s^{\omega}} \right)^{\ln R_n/\ln R_s} \]
    \[ \propto \left( \frac{R_n^{\omega}}{R_s^{\omega}} \right)^{\ln R_n/\ln R_s} \propto a_\omega^{R_n/\ln R_s} \Rightarrow h = \ln R_s/\ln R_n \]
Connecting exponents
Only 3 parameters are independent: e.g., take $d$, $R_n$, and $R_s$

<table>
<thead>
<tr>
<th>relation</th>
<th>scaling relation/parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell \sim L^d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$T_k = T_1(R_T)^{k-1}$</td>
<td>$T_1 = R_n = R_s - 2 + 2R_s/R_R$</td>
</tr>
<tr>
<td>$n_a/n_{a+1} = R_n$</td>
<td>$R_n$</td>
</tr>
<tr>
<td>$n_{a+1} - n_a = R_s$</td>
<td>$R_s = R_n$</td>
</tr>
<tr>
<td>$\ell \sim a^\gamma$</td>
<td>$h = \log(R_u)/\log(R_0)$</td>
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<tr>
<td>$\ell \sim L^D$</td>
<td>$D = d/h$</td>
</tr>
<tr>
<td>$L_{\perp} \sim L_H$</td>
<td>$H = d/h - 1$</td>
</tr>
<tr>
<td>$P(a) \sim a^{-\tau}$</td>
<td>$\tau = 2 - h$</td>
</tr>
<tr>
<td>$P(\ell) \sim \ell^{-\gamma}$</td>
<td>$\gamma = 1/h$</td>
</tr>
<tr>
<td>$\lambda \sim a^\beta$</td>
<td>$\beta = 1 + h$</td>
</tr>
<tr>
<td>$\lambda \sim L^\omega$</td>
<td>$\omega = d$</td>
</tr>
</tbody>
</table>

Fluctuations

Moving beyond the mean:
- Both Horton’s laws and Tokunaga’s law relate average properties, e.g.,
  \[ \bar{a}_u/\bar{a}_{u-1} = R_s \]
- Natural generalization to consideration relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness...

A toy model—Scheidegger’s model

Directed random networks \[\text{[11, 12]}\]

- $P(\perp) = P(\perp) = 1/2$
- Flow is directed downwards
- Useful and interesting test case—more later...

Generalizing Horton’s laws

- $\ell_u \propto (R_i)^\omega \Rightarrow N(\ell|\omega) = (R_u/R_i)^{-\omega} F(r/R_i)$
- $a_u \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_u^2)^{-\omega} F(a/R_u)$
- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

Equipmentapping reexamined:
Recall this story:

Mississippi basin partitioning

Amazon basin partitioning

Mississippi length distributions

Amazon length distributions

References
- Models
- Nutshell
- Tokunaga
- Horton
- Scaling relations

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Generalizing Horton’s laws

- How well does overall basin fit internal pattern?
  - Actual length = 4920 km (at 1 km res)
  - Predicted Mean length = 11100 km
  - Predicted Std dev = 5600 km
  - Actual length/Mean length = 44%
  - Okay.

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

<table>
<thead>
<tr>
<th>basin</th>
<th>l_0</th>
<th>l_0</th>
<th>σ_l</th>
<th>l_0/σ_l</th>
<th>l_0/σ_l</th>
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<td>4.92</td>
<td>11.10</td>
<td>5.60</td>
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<tr>
<td>Congo</td>
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<td>5.75</td>
<td>0.50</td>
<td>0.57</td>
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<td>2.37</td>
<td>1.74</td>
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<table>
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<th>σ_a/σ_ℓ₀</th>
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<td>0.14</td>
<td>0.49</td>
<td>0.42</td>
<td>0.28</td>
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</table>

Combining stream segments distributions:

- Stream segments sum to give main stream lengths
  \( l_\omega = \sum_{\mu=1}^{\mu} s_\mu \)
- \( P(l_\omega) \) is a convolution of distributions for the \( s_\mu \)

Next level up: Main stream length distributions must combine to give overall distribution for stream length

- \( P(l) \sim l^{-\gamma} \)
- Another round of convolutions

Nutshell: \( \xi \simeq 900 \) m.

Number and area distributions for the Scheidegger model \( P(n_\delta) \) versus \( P(a_\delta) \).
Generalizing Tokunaga’s law

Scheidegger:

- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using $R_s$

Generalizing Tokunaga’s law

Mississippi:

- Same data collapse for Mississippi...

Generalizing Tokunaga’s law

So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_{\mu} \left[ \frac{T_{\mu,\nu}}{(R_s)^{\mu-1}} \right]$$

where

$$P_{\mu}(z) = \frac{1}{\xi} e^{-z/\xi}.$$ 

$$P(s_{\nu}) \equiv P(T_{\mu,\nu})$$

- Exponentials arise from randomness.
- Look at joint probability $P(s_{\nu}, T_{\mu,\nu})$.

Generalizing Tokunaga’s law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments

Generalizing Tokunaga’s law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order $\mu$ stream segment terminating is constant:
  $$\tilde{p}_\mu \approx \frac{1}{(R_s)^{\mu-1}\xi_s}$$
- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- $\Rightarrow$ random spatial distribution of stream segments

Generalizing Tokunaga’s law

- Joint distribution for generalized version of Tokunaga’s law:
  $$P(s_{\nu}, T_{\mu,\nu}) = \tilde{p}_\mu \left( \frac{s_{\nu}}{T_{\mu,\nu}} \right) \rho_{\nu-1}^{T_{\mu,\nu}-1} (1 - \tilde{p}_\nu)$$

where

- $\rho_{\nu}$ = probability of absorbing an order $\nu$ stream
- $\tilde{p}_\mu$ = probability of an order $\mu$ stream terminating
- Approximation: depends on distance units of $s_{\nu}$
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.
Generalizing Tokunaga’s law

- Now deal with thing:
  \[ P(s_\mu, \mu) = \tilde{P}_\nu(\frac{S_\mu}{T_\mu}) \rho_T T_\nu (1 - \rho_T - \tilde{P}_\nu)^{\eta_T - 1} \]
- Set \((x, y) = (s_\mu, \mu)\) and \(q = 1 - \rho_T - \tilde{P}_\nu\), approximate liberally.
- Obtain
  \[ P(x, y) = Nx^{-1/2} [F(y/x)]^{-x} \]
  where
  \[ F(v) = \left(\frac{1 - v}{q}\right)^{-1} \left(\frac{v}{\rho}\right)^{-v}. \]

Generalizing Tokunaga’s law

- Checking form of \(P(s_\mu, \mu)\) works:
  Scheidegger:

Generalizing Tokunaga’s law

- Checking form of \(P(s_\mu, \mu)\) works:
  Mississippi:

Models

- Random subnetworks on a Bethe lattice
  - Dominant theoretical concept for several decades.
  - Bethe lattices are fun and tractable.
  - Led to idea of “Statistical inevitability” of river network statistics
  - But Bethe lattices unconnected with surfaces.
  - In fact, Bethe lattices \(\simeq\) infinite dimensional spaces (oops).
  - So let’s move on...
Branching Networks II

Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models
Nutshell
References

61 of 74

Scheidegger's model

Directed random networks \[11, 12\]

\[ P(\downarrow) = P(\uparrow) = 1/2 \]

Functional form of all scaling laws exhibited but exponents differ from real world \[15, 16, 14\]

A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks

Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. \[10\]

- Landscapes \( h(x) \) evolve such that energy dissipation \( \dot{\varepsilon} \) is minimized, where

\[
\dot{\varepsilon} \propto \int d\vec{r} (\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^h
\]

- Landscapes obtained numerically give exponents near that of real networks.

- But: numerical method used matters.

- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network \[8\]

Theoretical networks

Summary of universality classes:

<table>
<thead>
<tr>
<th>network</th>
<th>( h )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-convergent flow</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Directed random</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>Undirected random</td>
<td>5/8</td>
<td>5/4</td>
</tr>
<tr>
<td>Self-similar</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>OCN's (I)</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>OCN's (II)</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>OCN's (III)</td>
<td>3/5</td>
<td>1</td>
</tr>
<tr>
<td>Real rivers</td>
<td>0.5–0.7</td>
<td>1.0–1.2</td>
</tr>
</tbody>
</table>

\( h \Rightarrow \ell \propto a^h \) (Hack's law).

\( d \Rightarrow \ell \propto L_d^d \) (stream self-affinity).
Nutshell

Branching networks II Key Points:
- Horton’s laws and Tokunaga law all fit together.
- nb. for 2-d networks, these laws are ‘planform’ laws and ignore slope.
- Abundant scaling relations can be derived.
- Can take $R_n$, $R_\ell$, and $d$ as three independent parameters necessary to describe all 2-d branching networks.
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and $d$ are needed.
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.

References I


References II


References III


References IV


References V


References VI


References VII
