Introduction

Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.

Ex
Geomorphological networks

Definitions
- Drainage basin for a point \( p \) is the complete region of land from which overland flow drains through \( p \).
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
- Okay for large-scale networks...

Basic basin quantities: \( a, l, L_\parallel, L_\perp \):

- \( a \) = drainage basin area
- \( l \) = length of longest (main) stream (which may be fractal)
- \( L = L_\parallel \) = longitudinal length of basin
- \( L = L_\perp \) = width of basin

Allometry

- Isometry: dimensions scale linearly with each other.
- Allometry: dimensions scale nonlinearly.

Basin allometry

Allometric relationships:

- \( \ell \propto a^h \) 
  reportedly \( 0.5 < h < 0.7 \)
- \( \ell \propto L^d \) 
  reportedly \( 1.0 < d < 1.1 \)
- Basin allometry:
  \[ L_\perp \propto a^{h/d} \equiv a^{1/D} \]

'Laws'

- Hack's law (1957)\(^2\):
  \[ \ell \propto a^h \]
  reportedly \( 0.5 < h < 0.7 \)
- Scaling of main stream length with basin size:
  \[ \ell \propto L^d \]
  reportedly \( 1.0 < d < 1.1 \)
- Basin allometry:
  \[ L_\perp \propto a^{h/d} \equiv a^{1/D} \]

There are a few more 'laws'\(^1\):

Relation: Name or description:

- \( T_k = T_k(R_T)^k \) Tokunaga's law
- \( \ell \sim L^d \) self-affinity of single channels
- \( n_0/n_{n+1} = R_0 \) Horton's law of stream numbers
- \( L_{n+1}/L_n = R_1 \) Horton's law of main stream lengths
- \( \bar{a}_n/\bar{a}_{n+1} = R_0 \) Horton's law of basin areas
- \( \bar{s}_n/\bar{s}_{n+1} = R_0 \) Horton's law of stream segment lengths
- \( L_\perp \sim L^H \) scaling of basin widths
- \( P(a) \sim a^{-\gamma} \) probability of basin areas
- \( P(l) \sim l^{-\gamma} \) probability of stream lengths
- \( \ell \sim a^h \) Hack's law
- \( a \sim L^D \) scaling of basin areas
- \( \Lambda \sim a^d \) Langbein's law
- \( \lambda \sim L^2 \) variation of Langbein's law
Reported parameter values: [1]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Real networks:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_n$</td>
<td>3.0–5.0</td>
</tr>
<tr>
<td>$R_a$</td>
<td>3.0–6.0</td>
</tr>
<tr>
<td>$R_l = R_T$</td>
<td>1.5–3.0</td>
</tr>
<tr>
<td>$T_1$</td>
<td>1.0–1.5</td>
</tr>
<tr>
<td>$d$</td>
<td>1.1 ± 0.01</td>
</tr>
<tr>
<td>$D$</td>
<td>1.8 ± 0.1</td>
</tr>
<tr>
<td>$h$</td>
<td>0.50–0.70</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1.43 ± 0.05</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.8 ± 0.1</td>
</tr>
<tr>
<td>$H$</td>
<td>0.75–0.80</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.50–0.70</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>1.05 ± 0.05</td>
</tr>
</tbody>
</table>

Stream Ordering:

Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.

Kind of a mess...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out...

Stream Ordering:

Method for describing network architecture:

- Introduced by Horton (1945) [3]
- Modified by Strahler (1957) [6]
- Can be seen as iterative trimming of a network.
Stream Ordering:

Another way to define ordering:

- As before, label all source streams as order $\omega = 1$.
- Follow all labelled streams downstream.
- Whenever two streams of the same order ($\omega$) meet, the resulting stream has order incremented by 1 ($\omega + 1$).
- If streams of different orders $\omega_1$ and $\omega_2$ meet, then the resultant stream has order equal to the largest of the two.
- Simple rule:
  \[ \omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2} \]
  where $\delta$ is the Kronecker delta.

Stream Ordering:

One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ... but relationships based on ordering appear to be robust to resolution changes.

Stream Ordering:

Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand network architecture.

Horton’s laws

Self-similarity of river networks

- First quantified by Horton (1945)\(^3\), expanded by Schumm (1956)\(^5\)

Three laws:

- Horton’s law of stream numbers:
  \[ \frac{n_\omega}{n_{\omega+1}} = R_n > 1 \]
- Horton’s law of stream lengths:
  \[ \frac{\ell_{\omega+1}}{\ell_\omega} = R_\ell > 1 \]
- Horton’s law of basin areas:
  \[ \frac{a_{\omega+1}}{a_\omega} = R_a > 1 \]

Horton’s Ratios

Horton’s laws are defined by three ratios:

- $R_n$, $R_\ell$, and $R_a$.
- Horton’s laws describe exponential decay or growth:
  \[ n_\omega = \frac{n_{\omega-1}}{R_n} = \frac{n_{\omega-2}}{R_n^2} = \cdots = \frac{n_1}{R_n^{\omega-1}} = n_1 \ e^{-(\omega-1) \ln R_n} \]
Horton's laws

Similar story for area and length:

\[ \overline{a}_\omega = \overline{a}_1 \theta^{(\omega - 1)} \ln \overline{R}_\omega \]
\[ \overline{\ell}_\omega = \overline{\ell}_1 \theta^{(\omega - 1)} \ln \overline{R}_\omega \]

As stream order increases, number drops and area and length increase.

Horton's laws

A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.
- Averaging for stream lengths and areas is within basins.
- Horton's ratios go a long way to defining a branching network...
- But we need one other piece of information...

Horton's laws

A bonus law:

- Horton's law of stream segment lengths:

\[ \overline{s}_{\omega+1}/\overline{s}_\omega = R_s > 1 \]

- Can show that \( R_s = R_\ell \).

Insert question 2, assignment 2 (II)

Horton's laws in the real world:

Blood networks:

- Horton's laws hold for sections of cardiovascular networks.
- Measuring such networks is tricky and messy...
- Vessel diameters obey an analogous Horton's law.

Data from real blood networks

<table>
<thead>
<tr>
<th>Network</th>
<th>( R_0 )</th>
<th>( R_\ell^{-1} )</th>
<th>( R_s^{-1} )</th>
<th>( \ln \overline{R}_\ell/\overline{R}_s )</th>
<th>( \ln \overline{R}<em>\ell/\overline{R}</em>\ell )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>West et al.</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>1/2</td>
<td>1/3</td>
<td>3/4</td>
</tr>
<tr>
<td>rat (PAT)</td>
<td>2.76</td>
<td>1.58</td>
<td>1.60</td>
<td>0.45</td>
<td>0.46</td>
<td>0.73</td>
</tr>
<tr>
<td>cat (PAT)</td>
<td>3.67</td>
<td>1.71</td>
<td>1.78</td>
<td>0.41</td>
<td>0.44</td>
<td>0.79</td>
</tr>
<tr>
<td>(Turcotte et al.(^{10}))</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>dog (PAT)</td>
<td>3.69</td>
<td>1.67</td>
<td>1.52</td>
<td>0.39</td>
<td>0.32</td>
<td>0.90</td>
</tr>
<tr>
<td>pig (LCX)</td>
<td>3.57</td>
<td>1.89</td>
<td>2.20</td>
<td>0.50</td>
<td>0.62</td>
<td>0.62</td>
</tr>
<tr>
<td>pig (RCA)</td>
<td>3.50</td>
<td>1.81</td>
<td>2.12</td>
<td>0.47</td>
<td>0.60</td>
<td>0.65</td>
</tr>
<tr>
<td>pig (LAD)</td>
<td>3.51</td>
<td>1.84</td>
<td>2.02</td>
<td>0.49</td>
<td>0.56</td>
<td>0.65</td>
</tr>
<tr>
<td>human (PAT)</td>
<td>3.03</td>
<td>1.60</td>
<td>1.49</td>
<td>0.42</td>
<td>0.36</td>
<td>0.83</td>
</tr>
<tr>
<td>human (PAT)</td>
<td>3.36</td>
<td>1.56</td>
<td>1.49</td>
<td>0.37</td>
<td>0.33</td>
<td>0.94</td>
</tr>
</tbody>
</table>
Horton’s laws

Observations:

- Horton’s ratios vary:
  - $R_n$: 3.0–5.0
  - $R_a$: 3.0–6.0
  - $R_v$: 1.5–3.0

- No accepted explanation for these values.
- Horton’s laws tell us how quantities vary from level to level ...
- ... but they don’t explain how networks are structured.

Tokunaga’s law

Delving deeper into network architecture:

- Tokunaga (1968) identified a clearer picture of network structure [7, 8, 9]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.
- Tokunaga’s law is also a law of averages.

Network Architecture

Tokunaga’s law

- Property 1: Scale independence—depends only on difference between orders:
  \[ T_{\mu,\nu} = T_{\mu-\nu} \]

- Property 2: Number of side streams grows exponentially with difference in orders:
  \[ T_{\mu,\nu} = T_1(R_T)^{\mu-\nu-1} \]

- We usually write Tokunaga’s law as:
  \[ T_k = T_1(R_T)^{k-1} \]
  where $R_T \simeq 2$

The Mississippi

A Tokunaga graph:
Nutshell:

Branching networks I:
- Show remarkable self-similarity over many scales.
- There are many interrelated scaling laws.
- Horton-Strahler stream ordering gives one useful way of getting at the architecture of branching networks.
- Horton’s laws reveal self-similarity.
- Horton’s laws can be misinterpreted as suggesting a pure hierarchy.
- Tokunaga’s laws neatly describe network architecture.
- Branching networks exhibit a mixed hierarchical structure.
- Horton and Tokunaga can be connected analytically (next up).

References I


References II


References III


References IV