Outline

Lognormals
Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan

References
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Lognormals and friends

Lognormals
Empirical Confusability
Random Multiplicative Growth Model
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References
Alternative distributions

There are other ‘heavy-tailed’ distributions:

1. The Log-normal distribution (있다)

   \[ P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \]

2. Weibull distributions (있다)

   \[ P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-\left(\frac{x}{\lambda}\right)^\mu} dx \]

   CCDF = stretched exponential (있다).

3. Gamma distributions (있다), and more.
Alternative distributions

There are other ‘heavy-tailed’ distributions:

1. The Log-normal distribution (\(\mathcal{LN}\))

   \[
P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)
   \]

2. Weibull distributions (\(\mathcal{WE}\))

   \[
P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^\mu} dx
   \]

   CCDF = stretched exponential (\(\mathcal{SE}\)).

3. Gamma distributions (\(\mathcal{GA}\)), and more.
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1. The Log-normal distribution

   \[ P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \]

2. Weibull distributions

   \[ P(x)dx = \frac{k}{\lambda} \left( \frac{x}{\lambda} \right)^{\mu-1} e^{-\left( x/\lambda \right)^\mu} dx \]

   CCDF = stretched exponential.

3. Gamma distributions, and more.
lognormals

The lognormal distribution:

\[ P(x) = \frac{1}{x \sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \]

- In \( x \) is distributed according to a normal distribution with mean \( \mu \) and variance \( \sigma \).
- Appears in economics and biology where growth increments are distributed normally.
Standard form reveals the mean $\mu$ and variance $\sigma^2$ of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \quad \text{median}_{\text{lognormal}} = e^\mu,$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \quad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$

All moments of lognormals are finite.
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Lognormals

- Standard form reveals the mean $\mu$ and variance $\sigma^2$ of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi} \sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

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- All moments of lognormals are finite.
Derivation from a normal distribution

Take $Y$ as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp \left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

Transform according to $P(x)dx = P(y)dy$:

$$\frac{dy}{dx} = \frac{1}{x} \Rightarrow dy = \frac{dx}{x}$$

$$P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$
Derivation from a normal distribution

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\Rightarrow P(x) \, dx = \frac{1}{x\sqrt{2\pi\sigma}} \, \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \, dx
\]
Confusion between lognormals and pure power laws

Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$. 
Confusion

What’s happening:

\[
\ln P(x) = \ln \left\{ \frac{1}{x \sqrt{2\pi} \sigma} \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}
\]

\[
= \ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}
\]

\[
= -\frac{1}{2\sigma^2} (\ln x)^2 + \left( \frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.
\]

⇒ If \( \sigma^2 \gg 1 \) and \( \mu \),

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\ln P(x) \sim -\ln x - \text{const}
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\Rightarrow \text{If } \sigma^2 \gg 1 \text{ and } \mu,
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\ln P(x) \sim -\ln x + \text{const}
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\]
Confusion

- Expect -1 scaling to hold until \((\ln x)^2\) term becomes significant compared to \((\ln x)\).

This happens when (roughly)

\[
\frac{1}{2\sigma^2} (\ln x)^2 \approx 0.05 \left( \frac{\mu}{\sigma^2} - 1 \right) \ln x
\]

\[
\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e
\]

\[
\approx 0.05(\sigma^2 - \mu)
\]

\[\Rightarrow\] If you find a -1 exponent, you may have a lognormal distribution...
Lognormals and friends

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Generating lognormals:

Random multiplicative growth:

\[ x_{n+1} = r x_n \]

where \( r > 0 \) is a random growth variable

- (Shrinkage is allowed)
- In log space, growth is by addition:

\[ \ln x_{n+1} = \ln r + \ln x_n \]

\[ \Rightarrow \ln x_n \text{ is normally distributed} \]

\[ \Rightarrow x_n \text{ is lognormally distributed} \]
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- Gibrat\(^2\) (1931) uses preceding argument to explain lognormal distribution of firm sizes (\(\gamma \approx 1\)).
- But Robert Axtell\(^1\) (2001) shows a power law fits the data very well with \(\gamma = 2\), not \(\gamma = 1\) (!).
- Problem of data censusing (missing small firms).

One mechanistic piece in Gibrat's model seems okay empirically: Growth rate \(r\) appears to be independent of firm size.\(^1\).
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\[
\text{Freq} \propto (\text{size})^{-\gamma}
\]
\[
\gamma \simeq 2
\]

- One mechanistic piece in Gibrat’s model seems okay empirically: Growth rate $r$ appears to be independent of firm size. [1].
An explanation

- Axtel (mis?)cites Malcai et al.’s (1999) argument for why power laws appear with exponent $\gamma \approx 1$

- The set up: $N$ entities with size $x_i(t)$

- Generally:
  $$x_i(t + 1) = r x_i(t)$$

  where $r$ is drawn from some happy distribution

- Same as for lognormal but one extra piece.

- Each $x_i$ cannot drop too low with respect to the other sizes:
  $$x_i(t + 1) = \max(r x_i(t), c \langle x_i \rangle)$$
An explanation

- Axtel (mis?) cites Malcai et al.’s (1999) argument \[5\] for why power laws appear with exponent \( \gamma \approx 1 \).
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An explanation
Some math later... Insert question from assignment

Find $P(x) \sim x^{-\gamma}$

where $\gamma$ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{(c/N)^{\gamma - 1}}{(c/N)^{\gamma - 1} - (c/N)} - 1 \right]$$

$N = \text{total number of firms}.$

Now, if $c/N \ll 1$, $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{1}{(c/N)} \right]$.

Which gives $\gamma \sim 1 + \frac{1}{1 - c}$

Groovy... $c$ small $\Rightarrow \gamma \sim 2$
An explanation
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$$N = \frac{(\gamma - 2) \left[ (c/N)^{\gamma-1} - 1 \right]}{(\gamma - 1) \left[ (c/N)^{\gamma-1} - (c/N) \right]}$$

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➤

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►

Now, if $c/N \ll 1$,  

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ -\frac{1}{(c/N)} \right]$$

Which gives $\gamma \sim 1 + \frac{1}{1 - c}$

► Groovy... $c$ small $\Rightarrow \gamma \sim 2$
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Groovy... \( c \) small \( \Rightarrow \) \( \gamma \approx 2 \)
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- Which gives \( \gamma \sim 1 + \frac{1}{1 - c} \)

- Groovy... \( c \) small \( \Rightarrow \gamma \approx 2 \)
The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size $x_i$ to vary
- Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- Back to no bottom limit: each $x_i$ follows a lognormal
- Sizes are distributed as

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x \sqrt{2\pi t}} \exp \left( - \frac{(\ln x - \mu)^2}{2t} \right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)
- Now averaging different lognormal distributions.
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Lognormals and friends

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Averaging lognormals

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Some enjoyable suffering leads to:

\[ P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2} \]
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\[ P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln x/m)^2}} \]

- Depends on sign of \( \ln x/m \), i.e., whether \( x/m > 1 \) or \( x/m < 1 \).

- ‘Break in scaling’ (not uncommon)

- Double-Pareto distribution

- First noticed by Montroll and Shlesinger \([7, 8]\)

- Later: Huberman and Adamic \([3, 4]\): Number of pages per website
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▶ Take-home message: Be careful out there...
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