**Generalized Contagion**

**Principles of Complex Systems**

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**Outline**

**Generalized Model of Contagion**

**References**

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**Some (of many) issues**

- Disease models assume independence of infectious events.
- Threshold models only involve proportions: \( \frac{3}{10} = \frac{30}{100} \).
- Threshold models ignore exact sequence of influences.
- Threshold models assume immediate polling.
- Mean-field models neglect network structure.
- Network effects only part of story: media, advertising, direct marketing.

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**Generalized model—ingredients**

- Incorporate memory of a contagious element \([1, 2]\).
- Population of \( N \) individuals, each in state S, I, or R.
- Each individual randomly contacts another at each time step.
  - \( \phi(t) \) = fraction infected at time \( t \)
  - \( \phi \) = probability of contact with infected individual.
- With probability \( p \), contact with infective leads to an exposure.
- If exposed, individual receives a dose of size \( d \) drawn from distribution \( f \). Otherwise \( d = 0 \).

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**Generalized contagion model**

**References**

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**Basic questions about contagion**

- How many types of contagion are there?
- How can we categorize real-world contagions?
- Can we connect models of disease-like and social contagion?
**Generalized model—ingredients**

$I \Rightarrow R$
When $D_{t,i} < d^*_i$, individual $i$ recovers to state $R$ with probability $r$.

$R \Rightarrow S$
Once in state $R$, individuals become susceptible again with probability $1 - r$.

**A visual explanation**

**Generalized model—heterogeneity, $r = 1$**

Fixed point equation:

$$\phi^* = \frac{\sum_{k=1}^{T} \left( \frac{1}{k} \right) (1 - \rho^*)^{T-k} P_k}{\sum_{k=1}^{T} \left( \frac{1}{k} \right) (1 - \rho^*)^{T-k} T^k}$$

Expand around $\phi^* = 0$ to find Spread from single seed if

$$pP_1 T \geq 1$$

$$\Rightarrow \rho_c = \frac{1}{TP_1}$$

**Heterogeneous case**

**Example configuration:**

- Dose sizes are lognormally distributed with mean 1 and variance 0.433.
- Memory span: $T = 10$.
- Thresholds are uniformly set at
  1. $d_1 = 0.5$
  2. $d_2 = 1.5$
  3. $d_3 = 3$
- Spread of dose sizes matters, details are not important.

**Generalized model**

Important quantities:

$$P_k = \int_{d^*}^{\infty} d d^* g(d^*) P \left( \sum_{j=1}^{k} D_{t,j} \geq d^* \right) \text{ where } 1 \leq k \leq T.$$ 

$P_k$ = Probability that the threshold of a randomly selected individual will be exceeded by $k$ doses.

e.g., $P_1$ = Probability that one dose will exceed the threshold of a random individual = Fraction of most vulnerable individuals.

**Heterogeneous case—Three universal classes**

- **Epidemic threshold:** $P_1 > P_2/2, \rho_c = 1/(TP_1) < 1$
- **Vanishing critical mass:** $P_1 < P_2/2, \rho_c = 1/(TP_1) < 1$
- **Pure critical mass:** $P_1 < P_2/2, \rho_c = 1/(TP_1) > 1$
Calculations—Fixed points for $r < 1$, $d^* = 2$, and $T = 3$

F.P. Eq: $\phi^* = \Gamma(p, \phi^*; r) = \sum_{i=d^*}^{T} \phi^* (1 - \rho \phi^*)^{T-i}$.

$\Gamma(p, \phi^*; r) = (1-r)(\rho \phi^*)^2 + \sum_{m=1}^{\infty} (1-r)^m (\rho \phi^*)^2 (1-\rho \phi^*)^2 \times$ 

$\left[ \chi_{m-1} + \chi_{m-2} + 2p(1-\rho)(\chi_{m-3} + \rho(1-\rho)^2) \right]$.

where $\chi_m(p, \phi^*) = \sum_{k=0}^{[m/3]} \binom{m-2k}{k} (1-\rho \phi^*)^{m-k} (\rho \phi^*)^k$.

Hysteresis in vanishing critical mass models

II-III transition generalizes:

$$\rho_c = 1/P_1(T + \tau)$$

where $\tau = 1/r = \text{expected recovery time}$

SIS model

Now allow $r < 1$:

II-III transition generalizes: $\rho_c = 1/P_1(T + \tau)$

(I-III transition less pleasant analytically)

More complicated models

⇒ Due to heterogeneity in individual thresholds.
⇒ Same model classification holds: I, II, and III.

Discussion

▶ Memory is crucial ingredient.
▶ Three universal classes of contagion processes:
  I. Epidemic Threshold
  II. Vanishing Critical Mass
  III. Critical Mass

▶ Dramatic changes in behavior possible.
▶ To change kind of model: ‘adjust’ memory, recovery, fraction of vulnerable individuals ($T$, $r$, $\rho$, $P_1$, and/or $P_2$).
▶ To change behavior given model: ‘adjust’ probability of exposure ($p$) and/or initial number infected ($\phi_0$).
Discussion

- If \( p \geq 1 \), contagion can spread from single seed.
- Key quantity: \( p_c = 1 / [T + \tau] \)
- Depends only on:
  1. System Memory \( (T + \tau) \).
  2. Fraction of highly vulnerable individuals \( (P_i) \).
- Details unimportant (Universality):
  Many threshold and dose distributions give same \( P_k \).
- Most vulnerable/gullible population may be more important than small group of super-spreaders or influentials.

Future work/questions

- Do any real diseases work like this?
- Examine model’s behavior on networks
- Media/advertising + social networks model
- Classify real-world contagions

References I

Universal behavior in a generalized model of contagion.

A generalized model of social and biological contagion.