Fundamental Theorem of Linear Algebra

Now we see:
- Each of the four fundamental subspaces has a ‘best’ orthonormal basis
  - The \( \tilde{v}_i \) span \( \mathbb{R}^n \)
  - We find the \( \hat{v}_i \) as eigenvectors of \( A^T A \).
  - The \( \hat{u}_i \) span \( \mathbb{R}^m \)
  - We find the \( \hat{u}_i \) as eigenvectors of \( AA^T \).

Happy bases
- \( \{ \hat{v}_1, \ldots, \hat{v}_r \} \) span Row space
- \( \{ \hat{v}_{r+1}, \ldots, \hat{v}_n \} \) span Null space
- \( \{ \hat{u}_1, \ldots, \hat{u}_r \} \) span Column space
- \( \{ \hat{u}_{r+1}, \ldots, \hat{u}_m \} \) span Left Null space

Fundamental Theorem of Linear Algebra

How \( A \hat{x} \) works:
- \( A \hat{v}_i = \sigma_i \hat{u}_i \) for \( i = 1, \ldots, r \)
- \( A \hat{u}_i = 0 \) for \( i = r+1, \ldots, n \)
- Matrix version: 
  \[
  A = U \Sigma V^T
  \]
- \( A \) sends each \( \hat{u}_i \in C(A^T) \) to its partner \( \hat{v}_i \in C(A) \) with a positive stretch/shrink factor \( \sigma_i > 0 \).
- \( A \) is diagonal with respect to these bases.
- When viewed in the right way, every \( A \) is a diagonal matrix \( \Sigma \).

Fundamental Theorem of Linear Algebra

Outline
- The Fundamental Theorem of Linear Algebra
- Approximating matrices with SVD

Fundamental Theorem of Linear Algebra

- Applies to any \( m \times n \) matrix \( A \).
- Symmetry of \( A \) and \( A^T \).

Where \( \hat{x} \) lives:
- Row space \( C(A^T) \subset \mathbb{R}^n \).
- (Right) Nullspace \( N(A) \subset \mathbb{R}^n \).
- \( \dim C(A^T) + \dim N(A) = r + (n-r) = n \)
- Orthogonality: \( C(A^T) \perp N(A) = \mathbb{R}^m \)

Where \( \hat{b} \) lives:
- Column space \( C(A) \subset \mathbb{R}^m \).
- Left Nullspace \( N(A^T) \subset \mathbb{R}^m \).
- \( \dim C(A) + \dim N(A^T) = r + (m-r) = m \)
- Orthogonality: \( C(A) \perp N(A^T) = \mathbb{R}^m \)
The complete big picture:

Image approximation (80x60)

**Idea: use SVD to approximate images**

- Interpret elements of matrix $A$ as color values of an image.
- Truncate series SVD representation of $A$:
  $$A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T$$
- Use fact that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$.
- Rank $r = \min(m, n)$.
- Rank $r$ = # of pixels on shortest side (usually).
- For color: approximate 3 matrices (RGB).

From assignment 10

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$