Outline

The Fundamental Theorem of Linear Algebra

Approximating matrices with SVD
Fundamental Theorem of Linear Algebra

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^T$.

Where $\vec{x}$ lives:

- Row space $C(A^T) \subset \mathbb{R}^n$.
- (Right) Nullspace $N(A) \subset \mathbb{R}^n$.
- $\dim C(A^T) + \dim N(A) = r + (n - r) = n$
- Orthogonality: $C(A^T) \otimes N(A) = \mathbb{R}^n$

Where $\vec{b}$ lives:

- Column space $C(A) \subset \mathbb{R}^m$.
- Left Nullspace $N(A^T) \subset \mathbb{R}^m$.
- $\dim C(A) + \dim N(A^T) = r + (m - r) = m$
- Orthogonality: $C(A) \otimes N(A^T) = \mathbb{R}^m$
Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:
Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a ‘best’ orthonormal basis
- The $\hat{v}_i$ span $\mathbb{R}^n$
- We find the $\hat{v}_i$ as eigenvectors of $A^T A$.
- The $\hat{u}_i$ span $\mathbb{R}^m$
- We find the $\hat{u}_i$ as eigenvectors of $AA^T$.

Happy bases

- $\{\hat{v}_1, \ldots, \hat{v}_r\}$ span Row space
- $\{\hat{v}_{r+1}, \ldots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \ldots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$ span Left Null space
Fundamental Theorem of Linear Algebra

How $A\hat{x}$ works:

- $A\hat{v}_i = \sigma_i \hat{u}_i$ for $i = 1, \ldots, r$.
- $A\hat{v}_i = \hat{0}$ for $i = r + 1, \ldots, n$.

- Matrix version:
  
  $A = U\Sigma V^T$

- $A$ sends each $\hat{v}_i \in C(A^T)$ to its partner $\hat{u}_i \in C(A)$ with a positive stretch/shrink factor $\sigma_i > 0$.
- $A$ is diagonal with respect to these bases.
- When viewed in the right way, every $A$ is a diagonal matrix $\Sigma$. 
The complete big picture:

- **Row Space**
  - $A \vec{x}_r = \vec{p}$
  - $\hat{v}_1, \hat{v}_r, \hat{v}_{r+1}, \hat{v}_n$
  - $d = r$

- **Column Space**
  - $A \vec{x}_* = \vec{p}$
  - $\hat{u}_1, \hat{u}_r, \hat{u}_{r+1}, \hat{u}_m$
  - $d = r$

- **Null Space**
  - $A \vec{x}_n = \vec{0}$
  - $\vec{x}_n$
  - $d = n - r$

- **Left Null Space**
  - $\vec{0}$
  - $d = m - r$

- **SVD Equation**
  - $A \hat{v}_i = \sigma_i \hat{u}_i$
From assignment 10

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ A = U \Sigma V^T = \begin{bmatrix} 1 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{3}} & -1 & \frac{1}{\sqrt{3}} \end{bmatrix} \]
From assignment 10
Image approximation (80x60)

Idea: use SVD to approximate images

- Interpret elements of matrix $A$ as color values of an image.
- Truncate series SVD representation of $A$:

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T$$

- Use fact that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$.
- Rank $r = \min(m, n)$.
- Rank $r = \#$ of pixels on shortest side (usually).
- For color: approximate 3 matrices (RGB).