Chapter 3/4: Lecture 15
Linear Algebra
MATH 124, Fall, 2010

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Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:
  1. Big Picture of \( A\vec{x} = \vec{b} \)
     Must be able to draw the big picture!
  2. Projections and the normal equation
- As always, want ‘doing’ and ‘understanding’ and abilities.
Stuff to know/understand

**Vector Spaces:**

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a **spanning set** of vectors.
- Concept of a **basis**.
- Basis = minimal spanning set.
- Concept of **orthogonal complement**.
- Various techniques for finding bases and orthogonal complements.
Stuff to know/understand:

Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^T$.

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^T) \subset R^m$.

- $\dim C(A) + \dim N(A^T) = r + (m - r) = m$
- Orthogonality: $C(A) \otimes N(A^T) = R^m$

- Row space $C(A^T) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.

- $\dim C(A^T) + \dim N(A) = r + (n - r) = n$
- Orthogonality: $C(A^T) \otimes N(A) = R^n$
Stuff to know/understand:

**Finding four fundamental subspaces:**

- Enough to find bases for subspaces.
- Be able to reduce \( A \) to \( R \).
- Identify pivot columns and free columns.
- **Rank** \( r \) of \( A \) = # pivot columns.
- Know that relationship between \( R \)'s columns hold for \( A \)'s columns.
- **Warning:** \( R \)'s columns do not give a basis for \( C(A) \).
- But find pivot columns in \( R \), and same columns in \( A \) form a basis for \( C(A) \).
Stuff to know/understand:

More on bases for column and row space:

- Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
- Find conditions on $\vec{b}$’s elements for a solution to $A\vec{x} = \vec{b}$ to exist → obtain basis for $C(A)$.
- Basis for row space = non-zero rows in $R$ (easy!)
- Alternate basis for column space = non-zero rows in reduced form of $A^T$ (easy!)
Stuff to know/understand:

**Bases for nullspaces, left and right:**

- Basis for nullspace obtained by solving $A\vec{x} = \vec{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- # free variables $= n - \#$ pivot variables $= n - r = \dim N(A)$.
- Similarly find basis for $N(A^T)$ by solving $A^T\vec{y} = \vec{0}$. 
Stuff to know/understand:

Number of solutions to $A\vec{x} = \vec{b}$:

1. If $\vec{b} \not\in C(A)$, there are no solutions.

2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.
   - Number of solutions now depends entirely on $N(A)$.
   - If $\dim N(A) = n - r > 0$, then there are infinitely many solutions.
   - If $\dim N(A) = n - r = 0$, then there is one solution.
Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b} = \vec{p} + \vec{e}$
- $\vec{p}$ = that part of $\vec{b}$ that lies in the line:
  \[ \vec{p} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \vec{a} = \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} \vec{b} \]
- $\vec{e}$ = that part of $\vec{b}$ that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
- Understand construction and use of subspace projection operator $P$:
  \[ \vec{P} = A(A^T A)^{-1} A^T, \]
  where $A$'s columns form a subspace basis.
Stuff to know/understand

**Normal equation for** $A\vec{x} = \vec{b}$:

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
- Error vector must be orthogonal to column space so $A^T \vec{e} = A^T(\vec{b} - \vec{p}) = \vec{0}$.
- Rearrange:

  $$A^T \vec{p} = A^T \vec{b}$$

- Since $A\vec{x}_* = \vec{p}$, we end up with

  $$A^T A\vec{x}_* = A^T \vec{b}.$$ 

- This is linear algebra’s **normal equation**; $\vec{x}_*$ is our best solution to $A\vec{x} = \vec{b}$. 
The symmetry of \( A\vec{x} = \vec{b} \) and \( A^T\vec{y} = \vec{c} \):

- Row Space: \( C(A^T) \)
  - \( d = r \)
  - \( A^T\vec{y}_r = \vec{c} \)
  - \( \vec{0} \)
- Column Space: \( C(A) \)
  - \( d = r \)
  - \( A\vec{x}_r = \vec{b} \)
  - \( \vec{0} \)
- Null Space: \( N(A) \)
  - \( d = n - r \)
  - \( A^T\vec{y}_n = \vec{0} \)
  - \( \vec{0} \)
- Left Null Space: \( N(A^T) \)
  - \( d = m - r \)
  - \( A\vec{x}_n = \vec{0} \)
  - \( \vec{0} \)
How $A\vec{x} = \vec{b}$ works:

$A\vec{x}_r = \vec{b}$

$\vec{x} = \vec{x}_r + \vec{x}_n$

$A\vec{x}_n = \vec{0}$

$d = r$

$d = n - r$

$d = m - r$
Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:

- $R^n$: Row Space
- $R^m$: Column Space
- $\vec{0}$: Null Space
- $\vec{p}$: Best solution
- $\vec{e}$: Error
- $A\vec{x}_* = \vec{p}$
- $A\vec{x}_r = \vec{p}$
- $A\vec{x}_n = \vec{0}$
- $d = m - r$
- $d = n - r$
- $d = r$
- $\vec{x}_* = \vec{x}_r + \vec{x}_n$
The fourfold ways of $A\vec{x} = \vec{b}$:

<table>
<thead>
<tr>
<th>case</th>
<th>example $R$</th>
<th>big picture</th>
<th># solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = r$, $n = r$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td><img src="image1" alt="Diagram" /></td>
<td>1 always</td>
</tr>
<tr>
<td>$m = r$, $n &gt; r$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; _1 \ 0 &amp; 1 &amp; _2 \end{bmatrix}$</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$\infty$ always</td>
</tr>
<tr>
<td>$m &gt; r$, $n = r$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \ 0 &amp; 0 \end{bmatrix}$</td>
<td><img src="image3" alt="Diagram" /></td>
<td>0 or 1</td>
</tr>
<tr>
<td>$m &gt; r$, $n &gt; r$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; _1 \ 0 &amp; 1 &amp; _2 \ 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td><img src="image4" alt="Diagram" /></td>
<td>0 or $\infty$</td>
</tr>
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