Solving $A\vec{x} = \vec{b}$
Outline

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- We (people + computers) solve systems of linear equations by a systematic method of **Elimination** followed by **Back substitution**
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

  \[-x_1 + 3x_2 = 1 \]
  \[2x_1 + x_2 = 5 \]
Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.
2. Swap rows if needed to create an ‘upper triangular form’

   e.g.

   \[
   \begin{align*}
   2x_1 - x_2 &= -1 \\
   2x_1 - x_2 &= 3 \\
   x_2 &= -1 \\
   x_2 &= 3
   \end{align*}
   \]
Gaussian elimination:

Solve:

\[
\begin{align*}
2x_1 - 3x_2 &= 3 \\
4x_1 - 5x_2 + x_3 &= 7 \\
2x_1 - x_2 - 3x_3 &= 5
\end{align*}
\]
Gaussian elimination:

**Summary:**

Using *row operations*, we turned this problem:

\[
A\vec{x} = \vec{b} :
\begin{bmatrix}
2 & -3 & 0 \\
4 & -5 & 1 \\
2 & -1 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
7 \\
5
\end{bmatrix}
\]

into this problem:

\[
U\vec{x} = \vec{d} :
\begin{bmatrix}
2 & -3 & 0 \\
0 & 1 & 1 \\
0 & 0 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
1 \\
0
\end{bmatrix}
\]

and the latter is *easy to solve* using *back substitution*.
Gaussian elimination:

**Defn:**
The entries along $U$'s main diagonal are the **pivots** of $A$. (The pivots are hidden—elimination finds them.)

**Defn:**
A matrix with only zeros below the main diagonal is called **upper triangular**. A matrix with only zeros above the main diagonal is called **lower triangular**. We get from $A$ to $U$ and the latter is always upper triangular.

**Defn:**
**Singular** means a system has no unique solution.

- It may have no solutions or infinitely many solutions.
- Singular = archaic way of saying ‘messed up.’

**Truth:**
If at least one pivot is zero, the matrix will be **singular**. (but the reverse is not necessarily true).
Gaussian elimination:

**The one true method:**

- We simplify $A$ using elimination in the same way every time.
- Eliminate entries one column at a time, moving left to right, and down each column.

\[
\begin{align*}
X + X + X + X &= X \\
1 \downarrow + X + X + X &= X \\
2 \downarrow + 4 \downarrow + X + X &= X \\
3 \nearrow + 5 \rightarrow + 6 + X &= X
\end{align*}
\]
Gaussian elimination:

- To eliminate entry in row $i$ of $j$th column, subtract a multiple $\ell_{ij}$ of the $j$th row from $i$.
- For example:

\[\begin{align*}
2x_1 + 3x_2 - 2x_3 + x_4 &= 1 \\
x_1 - 7x_2 + 3x_3 + x_4 &= 1 \\
-x_1 - 3x_2 - x_3 + 5x_4 &= -2 \\
2x_1 + x_2 - 2x_3 + 2x_4 &= 0
\end{align*}\]

$\ell_{21} = 1/2$, $\ell_{31} = -1/2$, $\ell_{41} = ?$.

- **Note:** we cannot find $\ell_{32}$ etc., until we are finished with row 1. Pivots are hidden!

- **Note:** the denominator of each $\ell_{ij}$ multiplier is the pivot in the $j$th column.