The Small-World Phenomenon
Complex Networks, CSYS/MATH 303, Spring, 2010

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.
Outline

History

An online experiment

Previous theoretical work

An improved model

References
Some problems for people thinking about people?:

How are social networks structured?
- How do we define connections?
- How do we measure connections?
- (remote sensing, self-reporting)

What about the dynamics of social networks?
- How do social networks evolve?
- How do social movements begin?
- How does collective problem solving work?
- How is information transmitted through social networks?
Social Search

A small slice of the pie:

- **Q.** Can people pass messages between distant individuals using only their existing social connections?
- **A.** Apparently yes...

Handles:

- The Small World Phenomenon
- or “Six Degrees of Separation.”
The problem

Stanley Milgram et al., late 1960’s:

- Target person worked in Boston as a stockbroker.
- 296 senders from Boston and Omaha.
- 20% of senders reached target.
- average chain length $\approx 6.5$. 
The problem

Lengths of successful chains:

From Travers and Milgram (1969) in Sociometry: “An Experimental Study of the Small World Problem.”
The problem

Two features characterize a social ‘Small World’:

1. Short paths exist and
2. People are good at finding them.
Social Search

Milgram’s small world experiment with e-mail [2]

The SMALL WORLD project is an online experiment to test the idea that any two people in the world can be connected via 'six degrees of separation'.

Your objective is to get a message to a "target person", somewhere in the world, by forwarding the message to a friend of yours--someone who is "closer" to the target than you are. (If you happen to know the target, you can of course send it to them.)

If we have asked you to participate (you would have received a message from a friend of yours), you should continue the chain.

If you are just visiting us, sign up to start a new chain.
Social search—the Columbia experiment

- 60,000+ participants in 166 countries
- 18 targets in 13 countries including
  - a professor at an Ivy League university,
  - an archival inspector in Estonia,
  - a technology consultant in India,
  - a policeman in Australia,
  - and
  - a veterinarian in the Norwegian army.
- 24,000+ chains
Social search—the Columbia experiment

- Milgram’s participation rate was roughly 75%
- Email version: Approximately 37% participation rate.
- Probability of a chain of length 10 getting through:
  \[.37^{10} \approx 5 \times 10^{-5}\]
- \(\Rightarrow\) 384 completed chains (1.6% of all chains).
Social search—the Columbia experiment

- Motivation/Incentives/Perception matter.
- If target *seems* reachable
  ⇒ participation more likely.
- Small changes in attrition rates
  ⇒ large changes in completion rates
- e.g., ↓ 15% in attrition rate
  ⇒ ↑ 800% in completion rate
Social search—the Columbia experiment

Successful chains disproportionately used

- weak ties (Granovetter)
- professional ties (34% vs. 13%)
- ties originating at work/college
- target’s work (65% vs. 40%)

...and disproportionately avoided

- hubs (8% vs. 1%) (+ no evidence of funnels)
- family/friendship ties (60% vs. 83%)

Geography → Work
Social search—the Columbia experiment

Senders of successful messages showed little absolute dependency on

- age, gender
- country of residence
- income
- religion
- relationship to recipient

Range of completion rates for subpopulations: 30% to 40%
Nevertheless, some weak discrepancies do exist...

An above average connector:
Norwegian, secular male, aged 30-39, earning over $100K, with graduate level education working in mass media or science, who uses relatively weak ties to people they met in college or at work.

A below average connector:
Italian, Islamic or Christian female earning less than $2K, with elementary school education and retired, who uses strong ties to family members.
Social search—the Columbia experiment

Mildly bad for continuing chain:
choosing recipients because “they have lots of friends” or because they will “likely continue the chain.”

Why:

- Specificity important
- Successful links used relevant information. (e.g. connecting to someone who shares same profession as target.)
Social search—the Columbia experiment

Basic results:

- $\langle L \rangle \approx 4.05$ for all completed chains
- $L_* = \text{Estimated ‘true’ median chain length (zero attrition)}$
- Intra-country chains: $L_* = 5$
- Inter-country chains: $L_* = 7$
- All chains: $L_* \approx 7$
- Milgram: $L_* \approx 9$
Previous work—short paths

- Connected random networks have short average path lengths:
  \[ \langle d_{AB} \rangle \sim \log(N) \]
  
  \( N = \) population size,
  
  \( d_{AB} = \) distance between nodes \( A \) and \( B \).

- But: social networks aren’t random...
Previous work—short paths

Need “clustering” (your friends are likely to know each other):
Non-randomness gives clustering

\[ d_{AB} = 10 \rightarrow \text{too many long paths.} \]
Randomness + regularity

Now have $d_{AB} = 3$  \quad $\langle d \rangle$ decreases overall
Small-world networks


Small-world networks were found everywhere:

- neural network of C. elegans,
- semantic networks of languages,
- actor collaboration graph,
- food webs,
- social networks of comic book characters,...

Very weak requirements:

- local regularity + random short cuts
Toy model

Regular

Small-world

Random

$p = 0$  Increasing randomness  $p = 1$
The structural small-world property

\begin{figure}
\centering
\includegraphics[width=\textwidth]{small_world_property.png}
\caption{Graph showing the ratio of the clustering coefficient \( C(p)/C(0) \) and the average path length \( L(p)/L(0) \) as a function of \( p \).}
\end{figure}

\begin{align*}
C(p)/C(0) & \quad \text{Clustering coefficient ratio} \\
L(p)/L(0) & \quad \text{Average path length ratio}
\end{align*}
But are these short cuts findable?

No.

Nodes cannot find each other quickly with any local search method.
Previous work—finding short paths

- What can a local search method reasonably use?
- How to find things without a map?
- Need some measure of distance between friends and the target.

Some possible knowledge:

- Target’s identity
- Friends’ popularity
- Friends’ identities
- Where message has been
Previous work—finding short paths

“Navigation in a small world.”

Allowed to vary:

1. local search algorithm
   and
2. network structure.
Previous work—finding short paths

Kleinberg’s Network:

1. Start with regular d-dimensional cubic lattice.
2. Add local links so nodes know all nodes within a distance $q$.
3. Add $m$ short cuts per node.
4. Connect $i$ to $j$ with probability

$$p_{ij} \propto d_{ij}^{-\alpha}.$$ 

- $\alpha = 0$: random connections.
- $\alpha$ large: reinforce local connections.
- $\alpha = d$: same number of connections at all scales.
Previous work—finding short paths

Theoretical optimal search:

- “Greedy” algorithm.
- Same number of connections at all scales: \( \alpha = d \).

Search time grows slowly with system size (like \( \log^2 N \)).

But: social networks aren’t lattices plus links.
Previous work—finding short paths

- If networks have hubs can also search well: Adamic et al. (2001) \(^1\)

\[ P(k_i) \propto k_i^{-\gamma} \]

where \(k = \text{degree of node } i \) (number of friends).

- Basic idea: get to hubs first (airline networks).

- But: hubs in social networks are limited.
The problem

If there are no hubs and no underlying lattice, how can search be efficient?

Which friend of \( a \) is closest to the target \( b \)?

What does ‘closest’ mean?

What is ‘social distance’?
The model

One approach: incorporate **identity**. (See “Identity and Search in Social Networks.” Science, 2002, Watts, Dodds, and Newman [5])

**Identity is formed from attributes such as:**

- Geographic location
- Type of employment
- Religious beliefs
- Recreational activities.

**Groups** are formed by people with at least one similar attribute.

Attributes ⇔ Contexts ⇔ Interactions ⇔ Networks.
Social distance—Bipartite affiliation networks

[Diagram of bipartite affiliation networks with labels a, b, c, d, e in contexts and individuals]

[Diagram of unipartite network with labels a, b, c, d, e]
Social distance—Context distance

The Small-World Phenomenon

History
An online experiment
Previous theoretical work
An improved model
References

Social distance refers to the perceived distance between individuals based on their context or occupation. This diagram illustrates the concept with a tree structure:

- Occupation
  - Education
    - High school teacher
    - Kindergarten teacher
  - Health care
    - Nurse
    - Doctor

The diagram shows the social distance between different contexts and occupations, highlighting how these distances can be measured.
Distance between two individuals $x_{ij}$ is the height of lowest common ancestor.

$x_{ij} = 3$, $x_{ik} = 1$, $x_{iv} = 4$. 
The model

- Individuals are more likely to know each other the closer they are within a hierarchy.
- Construct $z$ connections for each node using

$$p_{ij} = c \exp\{-\alpha x_{ij}\}.$$ 

- $\alpha = 0$: random connections.
- $\alpha$ large: local connections.
Social distance—Generalized context space

(Blau & Schwartz, Simmel, Breiger)
The model

\[ \vec{v}_i = [1 \ 1 \ 1]^T, \quad \vec{v}_j = [8 \ 4 \ 1]^T \]

\[ x_{ij}^1 = 4, \quad x_{ij}^2 = 3, \quad x_{ij}^3 = 1. \]

Social distance:

\[ y_{ij} = \min_{h} x_{ij}^h. \]
The model

Triangle inequality doesn’t hold:

\[ y_{ik} = 4 > y_{ij} + y_{jk} = 1 + 1 = 2. \]
The model

- Individuals know the identity vectors of
  1. themselves,
  2. their friends,
  and
  3. the target.
- Individuals can estimate the social distance between their friends and the target.
- Use a greedy algorithm + allow searches to fail randomly.
The model-results—searchable networks

\[ \alpha = 0 \text{ versus } \alpha = 2 \text{ for } N \approx 10^5: \]

- \( q \geq r \)
- \( q < r \)
- \( r = 0.05 \)

\( q = \) probability an arbitrary message chain reaches a target.

- A few dimensions help.
- Searchability decreases as population increases.
- Precise form of hierarchy largely doesn’t matter.
The model-results

Milgram’s Nebraska-Boston data:

Model parameters:

- $N = 10^8$,
- $z = 300, g = 100$,
- $b = 10$,
- $\alpha = 1, H = 2$;
- $\langle L_{\text{model}} \rangle \simeq 6.7$
- $L_{\text{data}} \simeq 6.5$
Social search—Data

Adamic and Adar (2003)

- For HP Labs, found probability of connection as function of organization distance well fit by exponential distribution.
- Probability of connection as function of real distance $\propto 1/r$. 
Social Search—Real world uses

- Tags create identities for objects
- **Website tagging:** [http://www.del.icio.us](http://www.del.icio.us)
- (e.g., Wikipedia)
- **Photo tagging:** [http://www.flickr.com](http://www.flickr.com)
- Dynamic creation of metadata plus links between information objects.
- **Folksonomy:** collaborative creation of metadata
Social Search—Real world uses

Recommender systems:

- Amazon uses people’s actions to build effective connections between books.
- Conflict between ‘expert judgments’ and tagging of the hoi polloi.
Conclusions

- Bare networks are typically unsearchable.
- Paths are findable if nodes understand how network is formed.
- Importance of identity (interaction contexts).
- Improved social network models.
- Construction of peer-to-peer networks.
- Construction of searchable information databases.
References I

Search in power-law networks.

An experimental study of search in global social networks.

Navigation in a small world.

An experimental study of the small world problem.