Outline

Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Largest component

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- Average Component Size

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Random networks

Pure, abstract random networks:

- Consider set of all networks with $N$ labelled nodes and $m$ edges.
- Standard random network = randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.
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Some features:

▶ Number of possible edges:

\[ 0 \leq m \leq \binom{N}{2} = \frac{N(N - 1)}{2} \]

▶ Given \( m \) edges, there are \( \binom{N}{m} \) different possible networks.

▶ Crazy factorial explosion for \( 1 \ll m \ll \binom{N}{2} \).

▶ Limit of \( m = 0 \): empty graph.

▶ Limit of \( m = \binom{N}{2} \): complete or fully-connected graph.

▶ Real world: links are usually costly so real networks are almost always sparse.
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How to build standard random networks:

▶ Given $N$ and $m$.
▶ Two probabilistic methods (we’ll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability $p$.
   ➤ Useful for theoretical work.

2. Take $N$ nodes and add exactly $m$ links by selecting edges without replacement.
   ➤ Algorithm: Randomly choose a pair of nodes $i$ and $j$, $i \neq j$, and connect if unconnected; repeat until all $m$ edges are allocated.
   ➤ Best for adding relatively small numbers of links (most cases).
   ➤ 1 and 2 are effectively equivalent for large $N$. 
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A few more things:

▶ For method 1, # links is probabilistic:

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\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N - 1)
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▶ So the expected or average degree is

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\langle k \rangle = \frac{2 \langle m \rangle}{N} = \frac{2}{N} p \frac{1}{2} N(N - 1) = \frac{2}{N} p \frac{1}{2} N(N - 1) = p(N - 1).
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▶ If we keep \( \langle k \rangle \) constant then \( p \propto 1/N \to 0 \) as \( N \to \infty \).
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Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary $m$, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.
Random networks: examples

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entire network:

largest component:

\[ N = 500, \text{ number of edges } m = 100 \]
\[ \langle k \rangle = 0.4 \]
Random networks: examples

entire network:  
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\[ N = 500, \text{ number of edges } m = 200 \]
\[ \langle k \rangle = 0.8 \]
Random networks: examples

entire network:  
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$N = 500$, number of edges $m = 230$
average degree $\langle k \rangle = 0.92$
Random networks: examples

entire network: 

largest component: 

\[ N = 500, \text{ number of edges } m = 240 \]
\[ \text{average degree } \langle k \rangle = 0.96 \]
Random networks: examples

$N = 500$, number of edges $m = 250$
average degree $\langle k \rangle = 1$
Random networks: examples

entire network: largest component:

$N = 500$, number of edges $m = 260$
average degree $\langle k \rangle = 1.04$
Random networks: examples

entire network:  

largest component:  

\( N = 500 \), number of edges \( m = 280 \)
average degree \( \langle k \rangle = 1.12 \)
Random networks: examples

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\[ N = 500, \text{ number of edges } m = 300 \]
\[ \langle k \rangle = 1.2 \]
Random networks: examples

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\[ N = 500, \text{number of edges } m = 500 \]
\[ \langle k \rangle = 2 \]
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\[ \langle k \rangle = 4 \]
Random networks: examples for $N=500$

$m = 100 \quad \langle k \rangle = 0.4$

$m = 200 \quad \langle k \rangle = 0.8$

$m = 230 \quad \langle k \rangle = 0.92$

$m = 240 \quad \langle k \rangle = 0.96$

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Random networks: examples for $N=500$

$m = 250$
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Random networks: largest components

$\langle k \rangle = 1$

$m = 250$
Random networks

Clustering:

- For method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient (Newman \[1\]):
  \[ C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}} \]
- Recall: \( C_2 \) = probability that two nodes are connected given they have a friend in common.
- For standard random networks, we have simply that \( C_2 = p \).
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Clustering:

- So for large random networks \( (N \to \infty) \), clustering drops to zero.
- Key structural feature of random networks is that they locally look like branching networks (no loops).
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Degree distribution:

- Recall $p_k = \text{probability that a randomly selected node has degree } k$.
- Consider method 1 for constructing random networks: each possible link is realized with probability $p$.
- Now consider one node: there are ‘$N - 1$ choose $k$’ ways the node can be connected to $k$ of the other $N - 1$ nodes.
- Each connection occurs with probability $p$, each non-connection with probability $(1 - p)$.
- Therefore have a binomial distribution:

$$P(k; p, N) = \binom{N - 1}{k} p^k (1 - p)^{N-1-k}.$$
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Limiting form of $P(k; p, N)$:

- Our degree distribution:
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- What happens as $N \to \infty$?

- We must end up with the normal distribution right?

- If $p$ is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \sim pN \to \infty$.

- But we want to keep $\langle k \rangle$ fixed...

- So examine limit of $P(k; p, N)$ when $p \to 0$ and $N \to \infty$ with $\langle k \rangle = p(N - 1) = \text{constant}$.
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\[
P(k; p, N) = \binom{N-1}{k} \left( \frac{\langle k \rangle}{N-1} \right)^k \left( 1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k}
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= \frac{(N-1)!}{k!(N-1-k)!} \left( \frac{\langle k \rangle}{N-1} \right)^k \left( 1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k}
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Then simplify:

$$P(k; p, N) = \frac{(N - 1)!}{k!(N - 1 - k)!} \frac{\langle k \rangle^k}{(N - 1)^k} \left( 1 - \frac{\langle k \rangle}{N - 1} \right)^{N-1-k}$$

Further simplification:

$$= \frac{(N - 1)(N - 2) \cdots (N - k)}{k!} \frac{\langle k \rangle^k}{(N - 1)^k} \left( 1 - \frac{\langle k \rangle}{N - 1} \right)^{N-1-k}$$
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- We are now here:

$$P(k; p, N) \sim \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N - 1}\right)^{N-1-k}$$

- Now use the excellent result:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$ (Use l'Hôpital’s rule to prove.)

- Identifying $n = N - 1$ and $x = -\langle k \rangle$:

$$P(k; \langle k \rangle) \sim \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \left(1 - \frac{\langle k \rangle}{N - 1}\right)^{-k} \to \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}.$$  

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References
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- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution $P_k$.
- Also known as the configuration model \[^1\].
- Can generalize construction method from ER random networks.
- Assign each node a weight $w$ from some distribution $P_w$ and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$ 

- But we’ll be more interested in
  1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

- $N = 1000$.
- $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- Set $P_0 = 0$ (no isolated nodes).
- Vary exponent $\gamma$ between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.
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Random networks: examples for $N=1000$

- $\gamma = 2.1$, $\langle k \rangle = 3.448$
- $\gamma = 2.19$, $\langle k \rangle = 2.986$
- $\gamma = 2.28$, $\langle k \rangle = 2.306$
- $\gamma = 2.37$, $\langle k \rangle = 2.504$
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Random networks: largest components

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- Normalization: we must have

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\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1
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- Checking:

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\sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}
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- Mean degree: we must have

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▶ The variance of degree distributions for random networks turns out to be very important.
▶ Use calculation similar to one for finding $\langle k \rangle$ to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

▶ Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

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The edge-degree distribution:

- The degree distribution $P_k$ is fundamental for our description of many complex networks.
- Again: $P_k$ is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define $Q_k$ to be the probability the node at a random end of a randomly chosen edge has degree $k$.
- Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto kP_k$$

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- For random networks, $Q_k$ is also the probability that a friend (neighbor) of a random node has $k$ friends.

- Useful variant on $Q_k$:

  
  
  $R_k = \text{probability that a friend of a random node has } k \text{ other friends.}$

  
  
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  R_k = \frac{(k + 1)P_{k+1}}{\sum_{k' = 0} \left( (k' + 1)P_{k' + 1} \right)} = \frac{(k + 1)P_{k+1}}{\langle k \rangle}
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- For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$ 

- Therefore:

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- Again, neatness of results is a special property of the Poisson distribution.

- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...
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Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

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- Key: Average depends on the 1st and 2nd moments of \( P_k \) and not just the 1st moment.

- Three peculiarities:
  1. We might guess \( \langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1) \) but it’s actually \( \langle k(k - 1) \rangle \).
  2. If \( P_k \) has a large second moment, then \( \langle k_2 \rangle \) will be big.
     (e.g., in the case of a power-law distribution)
  3. Your friends are different to you...
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▶ Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2}\right) \geq \langle k \rangle$$

▶ So only if everyone has the same degree (variance $\sigma^2 = 0$) can a node be the same as its friends.

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(Big) Reason #2:

▶ $\langle k \rangle_R$ is key to understanding how well random networks are connected together.

▶ e.g., we’d like to know what’s the size of the largest component within a network.

▶ As $N \to \infty$, does our network have a giant component?

▶ Defn: Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.

▶ Defn: Giant component = component that comprises a non-zero fraction of a network as $N \to \infty$.

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Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.

- Equivalently, expect exponential growth in node number as we move out from a random node.

- All of this is the same as requiring $\langle k \rangle_R > 1$.

- Giant component condition (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- Again, see that the second moment is an essential part of the story.

- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$
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▶ Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.

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▶ Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.

▶ When $\langle k \rangle < 1$, all components are finite.

▶ Fine example of a continuous phase transition ($\square$).

▶ We say $\langle k \rangle = 1$ marks the critical point of the system.
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Random networks with skewed $P_k$:

- e.g., if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$ then

$$\langle k^2 \rangle = c \sum_{k=0}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=0}^{\infty} x^{2-\gamma} dx$$

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- So giant component always exists for these kinds of networks.

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Random networks with skewed $P_k$:

- e.g., if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$ then

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And how big is the largest component?

- Define $S_1$ as the size of the largest component.
- Consider an infinite ER random network with average degree $\langle k \rangle$.
- Let’s find $S_1$ with a back-of-the-envelope argument.
- Define $\delta$ as the probability that a randomly chosen node does not belong to the largest component.
- Simple connection: $\delta = 1 - S_1$.
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$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

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Carrying on:

\[ \delta = \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \]

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- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of $S_1$:
  $$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$  
- As $\langle k \rangle \to 0$, $S_1 \to 0$.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$.
- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S > 0$ when $\langle k \rangle > 1$.
- Really a transcritical bifurcation$^2$. 
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Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability $\delta$ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
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- We can do this but we need to enhance our toolkit with Generatingfunctionology...[^3]
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Generating functions

- **Idea:** Given a sequence \( a_0, a_1, a_2, \ldots, \) associate each element with a distinct function or other mathematical object.

- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

**Definition:**

- The generating function (g.f.) for a sequence \( \{a_n\} \) is

\[
F(x) = \sum_{n=0}^{\infty} a_n x^n.
\]

- Roughly: transforms a vector in \( \mathbb{R}^\infty \) into a function defined on \( \mathbb{R}^1 \).

- Related to Fourier, Laplace, Mellin, . . .
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Simple example

Rolling dice:

- \( p_k = \Pr(\text{throwing a } k) = 1/6 \) where \( k = 1, 2, \ldots, 6 \).

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F(x) = \sum_{k=1}^{6} p_k x^k = \frac{1}{6} (x + x^2 + x^3 + x^4 + x^5 + x^6).
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We’ll come back to this simple example as we derive various delicious properties of generating functions.
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Example

- Take a degree distribution with exponential decay:
  \[ P_k = ce^{-\lambda k} \]
  where \( c = 1 - e^{-\lambda} \).

- The generating function for this distribution is
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- Notice that \( F(1) = c/(1 - e^{-\lambda}) = 1 \).

- For probability distributions, we must always have \( F(1) = 1 \) since
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  \[ F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}. \]

- Notice that \( F(1) = c / (1 - e^{-\lambda}) = 1 \).

- For probability distributions, we must always have \( F(1) = 1 \) since
  \[ F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1. \]
Outline

Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Largest component

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- Average Component Size

References
Properties of generating functions

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\[
\langle k \rangle = \sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{\infty} kP_k x^{k-1} \bigg|_{x=1} = \frac{d}{dx} F(x) \bigg|_{x=1} = F'(1)
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- In general, many calculations become simple, if a little abstract.

- For our exponential example:

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F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.
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Useful pieces for probability distributions:

- **Normalization:**
  \[ F(1) = 1 \]

- **First moment:**
  \[ \langle k \rangle = F'(1) \]

- **Higher moments:**
  \[ \langle k^n \rangle = \left( x \frac{d}{dx} \right)^n F(x) \bigg|_{x=1} \]

- **kth element of sequence (general):**
  \[ P_k = \frac{1}{k!} \frac{d^k}{dx^k} F(x) \bigg|_{x=0} \]
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\[ \langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1. \]

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We first need the g.f. for \( R_k \).

We’ll now use this notation:

- \( F_P(x) \) is the g.f. for \( P_k \).
- \( F_R(x) \) is the g.f. for \( R_k \).

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Size distributions

To figure out the size of the largest component \(S_1\), we need more resolution on component sizes.

Definitions:

\[ \pi_n = \text{probability that a random node belongs to a finite component of size } n < \infty. \]
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Local-global connection:

\[ P_k, R_k \leftrightarrow \pi_n, \rho_n \]
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G.f.’s for component size distributions:

\[ F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \quad \text{and} \quad F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n \]

The largest component:

- Subtle key: \( F_\pi(1) \) is the probability that a node belongs to a finite component.
- Therefore: \( S_1 = 1 - F_\pi(1) \).

Our mission, which we accept:

- Find the four generating functions \( F_P, F_R, F_\pi, \) and \( F_\rho \).
Size distributions

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- Find the four generating functions \( F_P, F_R, F_\pi, \) and \( F_\rho \).
Size distributions

G.f.’s for component size distributions:

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F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \quad \text{and} \quad F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n
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Useful results we’ll need for g.f.’s

Sneaky Result 1:

- Consider two random variables $U$ and $V$ whose values may be $0, 1, 2, \ldots$
- Write probability distributions as $U_k$ and $V_k$ and g.f.’s as $F_U$ and $F_V$.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{U} V^{(i)}$$

with each $V^{(i)} \overset{d}{=} V$

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\[ x^k \text{ piece of } \left( \sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j \]

\[ = \sum_{j=0}^{\infty} U_j (F_V(x))^j \]

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Useful results we’ll need for g.f.’s

Sneaky Result 2:

- Start with a random variable $U$ with distribution $U_k$ $(k = 0, 1, 2, \ldots)$
- SR2: If a second random variable is defined as $V = U + 1$ then $F_V(x) = xF_U(x)$

Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k = x \sum_{j=0}^{\infty} U_j x^j = xF_U(x).$$
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Generalization of SR2:

1. If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

2. If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x) = x^{-i} \sum_{k=0}^{\infty} U_k x^k.$$
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Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

$\pi_n = \text{probability that a random node belongs to a finite component of size } n$

\[
\pi_n = \sum_{k=0}^{\infty} P_k \times \Pr \left( \text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1 \right)
\]

Therefore:

\[
F_\pi(x) = x \cdot F_\rho(\rho(x))
\]

Extra factor of $x$ accounts for random node itself.
Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

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Connecting generating functions

- **Goal**: figure out forms of the component generating functions, \( F_\pi \) and \( F_\rho \).
- \( \pi_n \) = probability that a random node belongs to a finite component of size \( n \)

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- $\pi_n$ = probability that a random node belongs to a finite component of size $n$

$$\pi_n = \sum_{k=0}^{\infty} P_k \times \Pr\left(\text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1\right)$$

Therefore:

$$F_\pi(x) = \frac{x}{SR2} F_P(F_\rho(x))$$

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- Therefore: $F_\pi(x) = x \cdot \underbrace{F_P(F_\rho(x))}_{\text{SR2}}$  
  \[\underbrace{\text{SR1}}_{\text{Extra factor of } x}\]

- Extra factor of $x$ accounts for random node itself.
Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

$\pi_n$ = probability that a random node belongs to a finite component of size $n$

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left( \text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1 \right)$$

Therefore:

$$F_\pi(x) = x \left( F_P \left( F_\rho(x) \right) \right)$$

Extra factor of $x$ accounts for random node itself.
Connecting generating functions

- $\rho_n =$ probability that a random link leads to a finite subcomponent of size $n$.

- Invoke one step of recursion: $\rho_n =$ probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$\rho_n = \sum_{k=0}^{\infty} R_k \times \Pr\left( \text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1 \right)$$

Therefore:

$$F_{\rho}(x) = x F_R \left( F_{\rho}(x) \right)$$

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Therefore:

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F_\rho(x) = x \frac{F_R (F_\rho(x))}{SR2} \frac{1}{SR1}
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Connecting generating functions

We now have two functional equations connecting our generating functions:

\[ F_\pi(x) = xF_P(F_\rho(x)) \quad \text{and} \quad F_\rho(x) = xF_R(F_\rho(x)) \]

Taking stock: We know \( F_P(x) \) and \( F_R(x) = F'_P(x)/F'_P(1) \).

We first untangle the second equation to find \( F_\rho \).

We can do this because it only involves \( F_\rho \) and \( F_R \).

The first equation then immediately gives us \( F_\pi \) in terms of \( F_\rho \) and \( F_R \).
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Component sizes

Remembering vaguely what we are doing:

Finding $F_\pi$ to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.

Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P(F_\rho(1)) \text{ and } F_\rho(1) = F_R(F_\rho(1))$$

Solve second equation numerically for $F_\rho(1)$.

Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$. 
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Example: Standard random graphs.

We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = \frac{F'_P(x)}{F'_P(1)} = \frac{e^{-\langle k \rangle (1-x)}}{e^{-\langle k \rangle (1-x')}} \bigg|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x) \quad \ldots \text{aha!}$$

RHS’s of our two equations are the same.

So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$

Why our dirty (but wrong) trick worked earlier...
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3. **RHS’s of our two equations are the same.**
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- \[ \therefore F_\pi(x) = xe^{-\langle k \rangle (1-F_\pi(x))} \]

- We’re first after \( S_1 = 1 - F_\pi(1) \) so set \( x = 1 \) and replace \( F_\pi(1) \) by \( 1 - S_1 \):

  \[ 1 - S_1 = e^{-\langle k \rangle S_1} \]

  Or: \( \langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1} \)

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Average component size

Next: find **average size** of finite components $\langle n \rangle$.

Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.

Try to avoid finding $F_\pi(x)$...

Starting from $F_\pi(x) = xF_P(F_\rho(x))$, we differentiate:

$$F'_\pi(x) = F_P(F_\rho(x)) + xF'_\rho(x)F'_P(F_\rho(x))$$

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Now set $x = 1$ in both equations.

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$$F'_\rho(x) = F_R(F_\rho(x)) + xF'_\rho(x)F'_R(F_\rho(x))$$

► Now set $x = 1$ in both equations.
► We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).
► Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$. 
Average component size

**Example:** Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:
  \[ F'_\pi(x) = F_P(F_\pi(x)) + xF'_\pi(x)F'_P(F_\pi(x)) \]

  Rearrange:
  \[ F'_\pi(x) = \frac{F_P(F_\pi(x))}{1 - xF'_P(F_\pi(x))} \]

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.
- Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

  End result: \[ \langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)} \]
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**Example:** Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F_\pi'(x) = F_P(F_\pi(x)) + xF_\pi'(x)F_P(F_\pi(x))$$

Rearrange:

$$F_\pi'(x) = \frac{F_P(F_\pi(x))}{1 - xF_P'(F_\pi(x))}$$

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- Our result for standard random networks:

\[
\langle n \rangle = F_\pi'(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}
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- Recall that \( \langle k \rangle = 1 \) is the critical value of average degree for standard random networks.

- Look at what happens when we increase \( \langle k \rangle \) to 1 from below.

- We have \( S_1 = 0 \) for all \( \langle k \rangle < 1 \) so

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\langle n \rangle = \frac{1}{1 - \langle k \rangle}
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- This blows up as \( \langle k \rangle \to 1 \).

- **Reason:** we have a power law distribution of component sizes at \( \langle k \rangle = 1 \).

- Typical critical point behavior....
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- Limits of $\langle k \rangle = 0$ and $\infty$ make sense for
  
  $$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}$.
- For $\langle k \rangle < 1$, $S_1 \sim \log N$. 
Average component size

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