Random walks and diffusion on networks
Complex Networks, CSYS/MATH 303, Spring, 2010

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Outline

Random walks on networks
Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node $j$ and take time to be discrete.
- Q: What’s the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\bar{p}(t)$.
- First task: connect $\bar{p}(t + 1)$ to $\bar{p}(t)$.
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- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
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- Let’s call our walker **Barry**.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is **hopelessly drunk**.
Where is Barry?

- Consider simple directed, ergodic (strongly connected) networks.
- As usual, represent network by adjacency matrix $A$ where

$$a_{ij} = 1 \text{ if } i \text{ has an edge leading to } j,$$

$$a_{ij} = 0 \text{ otherwise.}$$

- Barry is at node $i$ at time $t$ with probability $p_i(t)$.
- In the next time step he randomly lurches toward one of $i$’s neighbors.
- Equation-wise:

$$p_i(t + 1) = \sum_{j=1}^{n} \frac{1}{k_i} a_{ij} p_j(t).$$

where $k_i$ is $i$’s degree.
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- As usual, represent network by \textit{adjacency matrix} \( A \) where

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a_{ij} = \begin{cases} 
1 & \text{if } i \text{ has an edge leading to } j, \\
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- Barry is at node \( i \) at time \( t \) with probability \( p_i(t) \).
- In the next time step he \textit{randomly lurches} toward one of \( i \)'s neighbors.

- Equation-wise:

\[
p_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_i} a_{ji} p_j(t).
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where \( k_i \) is \( i \)'s degree. Note: \( k_i = \sum_{j=1}^{n} a_{ij} \).
Where is Barry?

- Linear algebra-based excitement:
  \[ p_i(t + 1) = \sum_{j=1}^{n} a_{ji} \frac{1}{k_j} p_j(t) \]
  is more usefully viewed as
  \[ \vec{p}(t + 1) = A^T K^{-1} \vec{p}(t) \]

  where \([K_{ij}] = [\delta_{ij} k_i]\) has node degrees on the main diagonal and zeros everywhere else.

- So... we need to find the dominant eigenvalue of \(A^T K^{-1}\).

- Expect this eigenvalue will be 1 (doesn’t make sense for total probability to change).

- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).

- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.
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- By inspection, we see that

$$\tilde{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \tilde{k}$$

satisfies $$\tilde{p}(\infty) = A^T K^{-1} \tilde{p}(\infty)$$ with eigenvalue 1.

- We will find Barry at node $i$ with probability proportional to its degree $k_i$.

- Nice implication: probability of finding Barry travelling along any edge is uniform.

- Diffusion in real space smooths things out.

- On networks, uniformity occurs on edges.

- So in fact, diffusion in real space is about the edges too but we just don’t see that.
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Other pieces:

- **Goodness:** $A^T K^{-1}$ is similar to a real symmetric matrix if $A = A^T$.

- Consider the transformation $M = K^{-1/2}$:

  $$K^{-1/2} A^T K^{-1} K^{1/2} = K^{-1/2} A^T K^{-1/2}.$$ 

- Since $A^T = A$, we have

  $$(K^{-1/2} A K^{-1/2})^T = K^{-1/2} A K^{-1/2}.$$ 

- **Upshot:** $A^T K^{-1} = A K^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.

- Can also show that maximum eigenvalue magnitude is indeed 1.

- Other goodies: next time round.
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