Random walks and diffusion on networks
Complex Networks, CSYS/MATH 303, Spring, 2010

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Outline

Random walks on networks

References
Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node $j$ and take time to be discrete.
- **Q:** What’s the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\bar{p}(t)$.
- First task: connect $\bar{p}(t + 1)$ to $\bar{p}(t)$.
- Let’s call our walker **Barry**.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is **hopelessly drunk**.
Where is Barry?

- Consider simple undirected networks with an edges either present of absent.
- Represent network by a symmetric adjacency matrix $A$ where
  
  $$a_{ij} = 1 \text{ if } i \text{ and } j \text{ are connected},$$
  
  $$a_{ij} = 0 \text{ otherwise}.$$

- Barry is at node $i$ at time $t$ with probability $p_i(t)$.
- In the next time step he randomly lurches toward one of $i$’s neighbors.
- Equation-wise:

  $$p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{k_i} a_{ji} p_i(t).$$

  where $k_i$ is $i$’s degree. Note: $k_i = \sum_{j=1}^{n} a_{ij}$. 
Where is Barry?

- Linear algebra-based excitement:
  \[ p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{k_i} a_{ji} p_i(t) \]
  is more usefully viewed as
  \[ \vec{p}(t + 1) = A^T K^{-1} \vec{p}(t) \]
  where \([K_{ij}] = [\delta_{ij} k_i] \)
  has node degrees on the main diagonal and zeros everywhere else.

- So... we need to find the dominant eigenvalue of
  \( A^T K^{-1} \).

- Expect this eigenvalue will be 1 (doesn’t make sense
  for total probability to change).

- The corresponding eigenvector will be the limiting
  probability distribution (or invariant measure).

- Extra concerns: multiplicity of eigenvalue = 1, and
  network connectedness.
Where is Barry?

- By inspection, we see that

\[ \vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k} \]

satisfies \( \vec{p}(\infty) = A^T K^{-1} \vec{p}(\infty) \) with eigenvalue 1.

- We will find Barry at node \( i \) with probability proportional to its degree \( k_i \).

- Nice implication: probability of finding Barry travelling along any edge is uniform.

- Diffusion in real space smooths things out.

- On networks, uniformity occurs on edges.

- So in fact, diffusion in real space is about the edges too but we just don’t see that.
Other pieces:

- Good news: $A^T K^{-1}$ is similar to a real symmetric matrix.
- Consider the transformation $M = K^{-1/2}$:
  \[
  K^{-1/2} A^T K^{-1} K^{1/2} = K^{-1/2} A^T K^{-1/2}.
  \]
- Since $A^T = A$, we have
  \[
  (K^{-1/2} A K^{-1/2})^T = K^{-1/2} A K^{-1/2}.
  \]
- Upshot: $A^T K^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.
- Other goodies: next time round.