Contagion

Complex Networks, CSYS/MATH 303, Spring, 2010

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Some large questions concerning network contagion:

1. For a given spreading mechanism on a given network, what’s the probability that there will be global spreading?
2. If spreading does take off, how far will it go?
3. How do the details of the network affect the outcome?
4. How do the details of the spreading mechanism affect the outcome?
5. What if the seed is one or many nodes?

Next up: We’ll look at some fundamental kinds of spreading on generalized random networks.
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- General spreading mechanism:
  State of node $i$ depends on history of $i$ and $i$’s neighbors’ states.

- Doses of entity may be stochastic and history-dependent.

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Spreading on Random Networks

- For random networks, we know local structure is pure branching.
- Successful spreading is contingent on single edges infecting nodes.

- Focus on binary case with edges and nodes either infected or not.
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Failure:

- Focus on binary case with edges and nodes either infected or not.
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Success

Failure:

- Focus on binary case with edges and nodes either infected or not.
Contagion condition

- We need to find:
  \[ r = \text{the average # of infected edges that one random infected edge brings about.} \]
- Define \( \beta_k \) as the probability that a node of degree \( k \) is infected by a single infected edge.

\[
\begin{align*}
  r &= \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \cdot \beta_k \cdot (k - 1) \\
  &= \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \cdot (1 - \beta_k) \cdot 0
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- Define \( \beta_k \) as the probability that a node of degree \( k \) is infected by a single infected edge.

\[ r = \sum_{k=0}^{\infty} \left( \frac{kP_k}{\langle k \rangle} \cdot \beta_k \cdot (k-1) \right) \]

\[ + \sum_{k=0}^{\infty} \left( \frac{kP_k}{\langle k \rangle} \cdot \left(1 - \beta_k\right) \cdot 0 \right) \]

where:
- \( kP_k \) is the probability of connecting to a degree \( k \) node.
- \( \langle k \rangle \) is the average degree of a node.
- \( \beta_k \) is the probability of a node of degree \( k \) being infected.
- \( (k-1) \) is the number of outgoing infected edges.
- \( 0 \) is the number of outgoing infected edges when there is no infection.
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Contagion condition

Our contagion condition is then:

$$r = \sum_{k=0}^{\infty} \frac{(k - 1)kP_k}{\langle k \rangle} \beta_k > 1.$$  

Case 1: If $\beta_k = 1$ then

$$r = \frac{\langle k(k - 1) \rangle}{\langle k \rangle} > 1.$$  

Good: This is just our giant component condition again.
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- **Case 2:** If $\beta_k = \beta < 1$ then

$$r = \beta \frac{\langle k(k - 1) \rangle}{\langle k \rangle} > 1.$$  

- A fraction $(1 - \beta)$ of edges do not transmit infection.
- Analogous phase transition to giant component case but critical value of $\langle k \rangle$ is increased.
- Aka bond percolation.
- Resulting degree distribution $P'_k$:

$$P'_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1 - \beta)^{i-k} P_i.$$  

- We can show $F_{P'}(x) = F_P(\beta x + 1 - \beta)$. 
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- **Cases 3, 4, 5, ...**: Now allow $\beta_k$ to depend on $k$
- **Asymmetry**: Transmission along an edge depends on node’s degree at other end.
- **Possibility**: $\beta_k$ increases with $k$... unlikely.
- **Possibility**: $\beta_k$ is not monotonic in $k$... unlikely.
- **Possibility**: $\beta_k$ decreases with $k$... hmmm.
- $\beta_k \downarrow$ is a plausible representation of a simple kind of social contagion.
- **The story**: More well connected people are harder to influence.
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- **Example:** $\beta_k = 1/k$.

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\begin{align*}
    r &= \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} \beta_k \\
    &= \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle k} \\
    &= \sum_{k=1}^{\infty} \frac{(k-1)P_k}{\langle k \rangle} = \frac{\langle k \rangle - 1}{\langle k \rangle} = 1 - \frac{1}{\langle k \rangle}
\end{align*}
\]

- Since $r$ is always less than 1, no spreading can occur for this mechanism.
- Decay of $\beta_k$ is too fast.
- Result is independent of degree distribution.
Contagion condition

Example: $\beta_k = 1/k$.

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Since $r$ is always less than 1, no spreading can occur for this mechanism.

Decay of $\beta_k$ is too fast.

Result is independent of degree distribution.
Contagion condition

- **Example:** $\beta_k = 1/k$.

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- Call these nodes vulnerables: they flip when only one of their friends flips.

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where $\left\lfloor \cdot \right\rfloor$ means floor.
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where $\lfloor \cdot \rfloor$ means floor.
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The contagion condition:

\[
\phi \left\lfloor \frac{1}{1} \right\rfloor \sum_{k=1}^{\langle k \rangle} \frac{(k - 1)kP_k}{\langle k \rangle} > 1.
\]

- As \( \phi \to 1 \), all nodes become resilient and \( r \to 0 \).
- As \( \phi \to 0 \), all nodes become vulnerable and the contagion condition matches up with the giant component condition.
- Key: If we fix \( \phi \) and then vary \( \langle k \rangle \), we may see two phase transitions.
- Added to our standard giant component transition, we will see a cut off in spreading as nodes become more connected.
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Basic Contagion Models

Social Contagion Models
   Network version
   All-to-all networks
   Theory

References
Social Contagion

Some important models (recap from CSYS 300)

- **Tipping models**—Schelling (1971)\(^{[8, 9, 10]}\)
  - Simulation on checker boards.
  - Idea of thresholds.
- **Threshold models**—Granovetter (1978)\(^{[7]}\)
- **Herding models**—Bikhchandani et al. (1992)\(^{[1, 2]}\)
  - Social learning theory, Informational cascades,...
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Original work:

“A simple model of global cascades on random networks”

- Mean field Granovetter model $\rightarrow$ network model
- Individuals now have a limited view of the world
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Threshold model on a network

- Interactions between individuals now represented by a network
  - Network is sparse
  - Individual $i$ has $k_i$ contacts
  - Influence on each link is reciprocal and of unit weight
  - Each individual $i$ has a fixed threshold $\phi_i$
  - Individuals repeatedly poll contacts on network
  - Synchronous, discrete time updating
  - Individual $i$ becomes active when fraction of active contacts $a_i \geq \phi_i k_i$
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- All nodes have threshold $\phi = 0.2$. 

$t=1$

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`Frame 17/58`
The most gullible

Vulnerables:

- Recall definition: individuals who can be activated by just one contact being active are **vulnerables**.
- The vulnerability condition for node $i$: $1/k_i \geq \phi_i$.
- Means # contacts $k_i \leq \lfloor 1/\phi_i \rfloor$.
- **Key:** For global cascades on random networks, must have a *global component of vulnerables*\(^{[12]}\).
- For a uniform threshold $\phi$, our contagion condition tells us when such a component exists:

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Cascades on random networks

- **Top curve:** final fraction infected if successful.
- **Middle curve:** chance of starting a global spreading event (cascade).
- **Bottom curve:** fractional size of vulnerable subcomponent.\(^{[12]}\)

\((\text{n.b., } z = \langle k \rangle)\)

- Cascades occur only if size of vulnerable subcomponent > 0.
- System is robust-yet-fragile just below upper boundary\(^{[3, 4, 11]}\)
- ‘Ignorance’ facilitates spreading.
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- Two phase transitions.

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- Now have endogenous mechanism for spreading from an individual to the critical mass and then beyond.
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Cascade window for random networks

( n.b., $z = \langle k \rangle$)

- Outline of cascade window for random networks.
Cascade window for random networks

\[ \Phi = \text{uniform individual threshold} \]

- Cascades
- No cascades

Example networks:
- Possible
- No Cascades
- Low influence

Influence vs. cascade size

\[ \langle S \rangle \]

References
Outline

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Granovetter’s Threshold model—recap

- Assumes deterministic response functions
  - $\phi_* = $ threshold of an individual.
  - $f(\phi_*) = $ distribution of thresholds in a population.
  - $F(\phi_*) = $ cumulative distribution = $\int_{\phi_*=0}^{\phi_*} f(\phi_') d\phi_*$
- $\phi_t = $ fraction of people ‘rioting’ at time step $t$. 

Graph showing the probability of activation against threshold $\phi$. The graph is a step function with a vertical jump at $\phi = 0.6$. The x-axis represents $\phi$ and the y-axis represents the probability of activation.
Social Contagion

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Social Sciences—Threshold models

- At time $t + 1$, fraction rioting = fraction with $\phi_* \leq \phi_t$.

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⇒ Iterative maps of the unit interval $[0, 1]$. 
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$\Rightarrow$ Iterative maps of the unit interval $[0, 1]$. 
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$$\phi_{t+1} = \int_0^{\phi_t} f(\phi_*)d\phi_* = F(\phi_*)\big|_0^{\phi_t} = F(\phi_t)$$

⇒ Iterative maps of the unit interval $[0, 1]$. 
Social Sciences—Threshold models

Action based on perceived behavior of others.

- Two states: S and I
- Recover now possible (SIS)
- $\phi = \text{fraction of contacts ‘on’ (e.g., rioting)}$
- Discrete time, synchronous update (strong assumption!)
- This is a Critical mass model
Social Sciences—Threshold models

Action based on perceived behavior of others.

- Two states: S and I
- Recover now possible (SIS)
- $\phi = \text{fraction of contacts ‘on’ (e.g., rioting)}$
- Discrete time, synchronous update (strong assumption!)
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Social Sciences—Threshold models

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Example of single stable state model
Social Sciences—Threshold models

Implications for collective action theory:

1. Collective uniformity \( \not\Rightarrow \) individual uniformity
2. Small individual changes \( \Rightarrow \) large global changes

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- Connect mean-field model to network model.
- Single seed for network model: \( 1/N \to 0 \).
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All-to-all versus random networks

A: all-to-all networks
B: random networks
C: all-to-all networks
D: random networks

\[ F(a_{t+1}) \]

\[ \langle S \rangle \]

\[ \langle k \rangle \]
Threshold contagion on random networks

Three key pieces to describe analytically:

1. The fractional size of the largest subcomponent of vulnerable nodes, $S_{vuln}$.
2. The chance of starting a global spreading event, $P_{trig} = S_{trig}$.
3. The expected final size of any successful spread, $S$.
   ▶ n.b., the distribution of $S$ is almost always bimodal.
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- **First goal:** Find the largest component of vulnerable nodes.
  - Recall that for finding the giant component’s size, we had to solve:
    \[
    F_\pi(x) = xF_p(F_\rho(x)) \quad \text{and} \quad F_\rho(x) = xF_R(F_\rho(x))
    \]
  - We’ll find a similar result for the subset of nodes that are vulnerable.
  - This is a node-based percolation problem.
  - For a general monotonic threshold distribution \( f(\phi) \), a degree \( k \) node is vulnerable with probability
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Everything now revolves around the modified generating function:

\[ F^{(vuln)}_P(x) = \sum_{k=0}^{\infty} \beta_k P_k x^k. \]

Generating function for friends-of-friends distribution is related in same way as before:

\[ F^{(vuln)}_R(x) = \frac{\frac{d}{dx} F^{(vuln)}_P(x)}{\frac{d}{dx} F^{(vuln)}_P(x)|_{x=1}}. \]
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- Functional relations for component size g.f.'s are almost the same...

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F^{(\text{vuln})}_\pi(x) = 1 - F^{(\text{vuln})}_P(1) + xF^{(\text{vuln})}_P(F^{(\text{vuln})}_\rho(x))
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- Central node is not vulnerable

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Assumption is first node is randomly chosen.

Same set up as for vulnerable component except now we don’t care if the initial node is vulnerable or not:

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Threshold contagion on random networks

- Third goal: Find expected fractional size of spread.
- Not obvious even for uniform threshold problem.
- Difficulty is in figuring out if and when nodes that need $\geq 2$ hits switch on.
- Problem solved for infinite seed case by Gleeson and Cahalane:
Third goal: Find expected fractional size of spread.

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Expected size of spread

Idea:

- Randomly turn on a fraction $\phi_0$ of nodes at time $t = 0$
- Capitalize on local branching network structure of random networks (again)
- Now think about what must happen for a specific node $i$ to become active at time $t$:
  - $t = 0$: $i$ is one of the seeds (prob = $\phi_0$)
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  - $t = 2$: enough of $i$’s friends and friends-of-friends switched on at time $t = 0$ so that $i$’s threshold is now exceeded.
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\( \bullet = \text{active, } \varphi = 1/3 \)

\( t=0 \)
Expected size of spread

\[ \phi = 1/3 \]

\( t = 1 \)

\( \bullet = \text{active} \)
Expected size of spread

\[
\text{= active, } \varphi = 1/3
\]

\[t=2\]
Expected size of spread

\[ \varphi = 1/3 \]

\[ t=3 \]

\[ = \text{active} \]
Expected size of spread

\[ \phi = \frac{1}{3}, \quad t = 4 \]

\[ i \]

\[ = \text{active} \]
Expected size of spread

$t=0$

$\varphi = 1/3$

- Red = active at $t=0$
- Yellow = active at $t=1$
- Green = active at $t=2$
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Frame 39/58
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Notes:

- Calculations are possible nodes do not become inactive.
- Not just for threshold model—works for a wide range of contagion processes.
- We can analytically determine the entire time evolution, not just the final size.
- We can in fact determine $\Pr(\text{node of degree } k \text{ switches on at time } t)$.
- Asynchronous updating can be handled too.
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- **Notation:** \( \Pr(\text{node } i \text{ becomes active at time } t) = \phi_{i,t} \).
- **Notation:** \( \beta_{kj} = \Pr(\text{a degree } k \text{ node becomes active if } j \text{ neighbors are active}) \).
- Our starting point: \( \phi_{i,0} = \phi_0 \).
- \( \binom{k_i}{j} \phi_0^j (1 - \phi_0)^{k_i-j} = \Pr(\text{j of node } i\text{'s } k_i \text{ neighbors were seeded at time } t = 0) \).
- Probability node \( i \) was a seed at \( t = 0 \) is \( \phi_0 \) (as above).
- Probability node \( i \) was not a seed at \( t = 0 \) is \( 1 - \phi_0 \).
- Combining everything, we have:

\[
\phi_{i,1} = \phi_0 + (1 - \phi_0) \sum_{j=0}^{k_i} \binom{k_i}{j} \phi_0^j (1 - \phi_0)^{k_i-j} \beta_{kij}.
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- **Notation:** $\Pr(\text{node } i \text{ becomes active at time } t) = \phi_{i,t}$.
- **Notation:** $\beta_{kj} = \Pr$ (a degree $k$ node becomes active if $j$ neighbors are active).
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First connect $\theta_0$ to $\theta_1$:

$\theta_1 = \phi_0 +$

$$(1 - \phi_0) \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_0^j (1 - \theta_0)^{k-1-j} \beta_{kj}$$

$\frac{kP_k}{\langle k \rangle} = R_k = \text{Pr}$ (edge connects to a degree $k$ node).

$\sum_{j=0}^{k-1}$ piece gives $\text{Pr}(\text{degree node } k \text{ activates})$ of its neighbors $k - 1$ incoming neighbors are active.

$\phi_0$ and $(1 - \phi_0)$ terms account for state of node at time $t = 0$.

See this all generalizes to give $\theta_{t+1}$ in terms of $\theta_t$...
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See this all generalizes to give $\theta_{t+1}$ in terms of $\theta_t$...
Expected size of spread

Two pieces:

1. $\theta_{t+1} = \phi_0 +
   \left(1 - \phi_0\right) \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_t^j (1 - \theta_t)^{k-1-j} \beta_{kj}
   
   with $\theta_0 = \phi_0$.

2. $\phi_{t+1} = \phi_0 +
   \left(1 - \phi_0\right) \sum_{k=0}^{\infty} P_k \sum_{j=0}^{k} \binom{k}{j} \theta_t^j (1 - \theta_t)^{k-j} \beta_{kj}.$
Comparison between theory and simulations

- Pure random networks with simple threshold responses
- \( R = \) uniform threshold (our \( \phi_\ast \)); \( z = \) average degree; \( \rho = \phi; q = \theta; N = 10^5 \).
- \( \phi_0 = 10^{-3}, 0.5 \times 10^{-2}, \) and \( 10^{-2} \).
- Cascade window is for \( \phi = 10^{-2} \) case.
- Sensible expansion of cascade window as \( \phi_0 \) increases.

From Gleeson and Cahalane [6]
Notes:

- Retrieve cascade condition for spreading from a single seed in limit $\phi_0 \to 0$.
- Depends on map $\theta_{t+1} = G(\theta_t; \phi_0)$.
- First: if self-starters are present, some activation is assured:

$$G(0; \phi_0) = \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \beta_{k0} > 0.$$ 

meaning $\beta_{k0} > 0$ for at least one value of $k \geq 1$.

- If $\theta = 0$ is a fixed point of $G$ (i.e., $G(0; \phi_0) = 0$) then spreading occurs if

$$G'(0; \phi_0) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k - 1)kP_k \beta_{k1} > 1.$$ 

Insert question from assignment 6 (⊞)
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Insert question from assignment 6 ( UINavigationController)
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Insert question from assignment 6 (⮩)
Notes:

In words:

- If $G(0; \phi_0) > 0$, spreading must occur because some nodes turn on for free.
- If $G$ has an unstable fixed point at $\theta = 0$, then cascades are also always possible.

Non-vanishing seed case:

- Cascade condition is more complicated for $\phi_0 > 0$.
- If $G$ has a stable fixed point at $\theta = 0$, and an unstable fixed point for some $0 < \theta_* < 1$, then for $\theta_0 > \theta_*$, spreading takes off.
- Tricky point: $G$ depends on $\phi_0$, so as we change $\phi_0$, we also change $G$. 
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Given $\theta_0 (= \phi_0)$, $\theta_\infty$ will be the nearest stable fixed point, either above or below.

- n.b., adjacent fixed points must have opposite stability types.
- Important: Actual form of $G$ depends on $\phi_0$.
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Comparison between theory and simulations

Now allow thresholds to be distributed according to a Gaussian with mean $R$.

- $R = 0.2, 0.362, \text{ and } 0.38; \sigma = 0.2$.

- $\phi_0 = 0$ but some nodes have thresholds $\leq 0$ so effectively $\phi_0 > 0$.

- Now see a (nasty) discontinuous phase transition for low $\langle k \rangle$.

From Gleeson and Cahalane [6]
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- n.b.: 0 is not a fixed point here: $\theta_0 = 0$ always takes off.
- Top to bottom: $R = 0.35, 0.371,$ and $0.375$.
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- Consider largest vulnerable component as initial set of seeds.
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Two pieces modified for single seed:

1. \( \theta_{t+1} = \theta_{\text{vuln}} + \)

\[
(1 - \theta_{\text{vuln}}) \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_t^j (1 - \theta_t)^{k-1-j} \beta_{kj}
\]

with \( \theta_0 = \theta_{\text{vuln}} = \text{Pr an edge leads to the giant vulnerable component (if it exists).} \)

2. \( \phi_{t+1} = S_{\text{vuln}} + \)

\[
(1 - S_{\text{vuln}}) \sum_{k=0}^{\infty} P_k \sum_{j=0}^{k} \binom{k}{j} \theta_t^j (1 - \theta_t)^{k-j} \beta_{kj}.
\]
Time-dependent solutions

Synchronous update

- Done: Evolution of $\phi_t$ and $\theta_t$ given exactly by the maps we have derived.

Asynchronous updates

- Update nodes with probability $\alpha$.
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- More on this later...
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References


References II

Highly optimized tolerance: Robustness and design in complex systems.

Cascades on correlated and modular random networks.

Seed size strongly affects cascades on random networks.

Threshold models of collective behavior.
References III

Dynamic models of segregation.

Hockey helmets, concealed weapons, and daylight saving: A study of binary choices with externalities.

*Micromotives and Macrobehavior.*

*Critical Phenomena in Natural Sciences.*
A simple model of global cascades on random networks.  
_pdf (.TabStop)