Branching Networks I
Complex Networks, CSYS/MATH 303, Spring, 2010

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

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Introduction

Branching networks are useful things:

- **Fundamental to material supply and collection**
  - **Supply**: From one source to many sinks in 2- or 3-d.
  - **Collection**: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

Examples:

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

Frame 3/38
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[1] Title: Branching Networks I
[2] Section: Introduction
[3] Subsection: Definitions
[5] Subsection: Laws
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[9] Subsection: Nutshell
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Branching networks are everywhere...

http://hydrosheds.cr.usgs.gov/
Branching networks are everywhere...

Geomorphological networks

Definitions

- **Drainage basin** for a point $p$ is the complete region of land from which overland flow drains through $p$.
- Definition most sensible for a point in a stream.
- **Recursive structure:** Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
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Basic basin quantities: $a$, $l$, $L_\parallel$, $L_\perp$:

- $a$ = drainage basin area
- $l$ = length of longest (main) stream (which may be fractal)
- $L_\parallel$ = longitudinal length of basin
- $L_\perp$ = width of basin

\[ L_\parallel = L \]
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Isometry: dimensions scale linearly with each other.
Allometry

Isometry: dimensions scale linearly with each other.

Allometry: dimensions scale nonlinearly.
Basin allometry

Allometric relationships:

- \( \ell \propto a^h \)
- \( \ell \propto L^d \)
- Combine above:
  \[ a \propto L^{d/h} \equiv L^D \]
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‘Laws’

- Hack’s law (1957) \(^2\):
  \[ l \propto a^h \]
  reportedly \(0.5 < h < 0.7\)

- Scaling of main stream length with basin size:
  \[ l \propto L_d \]
  reportedly \(1.0 < d < 1.1\)

- Basin allometry:
  \[ L_\parallel \propto a^{h/d} \equiv a^{1/D} \]
  \(D < 2 \rightarrow \) basins elongate.
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There are a few more ‘laws’: [1]

<table>
<thead>
<tr>
<th>Relation</th>
<th>Name or description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_k = T_1(R_T)^k$</td>
<td>Tokunaga’s law</td>
</tr>
<tr>
<td>$\ell \sim L^d$</td>
<td>self-affinity of single channels</td>
</tr>
<tr>
<td>$n_\omega / n_{\omega+1} = R_n$</td>
<td>Horton’s law of stream numbers</td>
</tr>
<tr>
<td>$\bar{\ell}<em>{\omega+1}/\bar{\ell}</em>\omega = R_\ell$</td>
<td>Horton’s law of main stream lengths</td>
</tr>
<tr>
<td>$\bar{a}<em>{\omega+1}/\bar{a}</em>\omega = R_a$</td>
<td>Horton’s law of basin areas</td>
</tr>
<tr>
<td>$\bar{s}<em>{\omega+1}/\bar{s}</em>\omega = R_s$</td>
<td>Horton’s law of stream segment lengths</td>
</tr>
<tr>
<td>$L_\perp \sim L^H$</td>
<td>scaling of basin widths</td>
</tr>
<tr>
<td>$P(a) \sim a^{-\tau}$</td>
<td>probability of basin areas</td>
</tr>
<tr>
<td>$P(\ell) \sim \ell^{-\gamma}$</td>
<td>probability of stream lengths</td>
</tr>
<tr>
<td>$\ell \sim a^h$</td>
<td>Hack’s law</td>
</tr>
<tr>
<td>$a \sim L^D$</td>
<td>scaling of basin areas</td>
</tr>
<tr>
<td>$\Lambda \sim a^{\beta}$</td>
<td>Langbein’s law</td>
</tr>
<tr>
<td>$\lambda \sim L^\varphi$</td>
<td>variation of Langbein’s law</td>
</tr>
</tbody>
</table>
**Reported parameter values:** \[1\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Real networks:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_n$</td>
<td>3.0–5.0</td>
</tr>
<tr>
<td>$R_a$</td>
<td>3.0–6.0</td>
</tr>
<tr>
<td>$R_\ell = R_T$</td>
<td>1.5–3.0</td>
</tr>
<tr>
<td>$T_1$</td>
<td>1.0–1.5</td>
</tr>
<tr>
<td>$d$</td>
<td>$1.1 \pm 0.01$</td>
</tr>
<tr>
<td>$D$</td>
<td>$1.8 \pm 0.1$</td>
</tr>
<tr>
<td>$h$</td>
<td>0.50–0.70</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$1.43 \pm 0.05$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$1.8 \pm 0.1$</td>
</tr>
<tr>
<td>$H$</td>
<td>0.75–0.80</td>
</tr>
<tr>
<td>$\beta$</td>
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<td>$1.05 \pm 0.05$</td>
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Kind of a mess...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values
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For (3): Many attempts: not yet sorted out...
Stream Ordering:

Method for describing network architecture:

- Introduced by Horton (1945) \(^3\)
- Modified by Strahler (1957) \(^6\)
- Term: Horton-Strahler Stream Ordering \(^4\)
- Can be seen as iterative trimming of a network.
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Stream Ordering:

Some definitions:

▷ A **channel head** is a point in landscape where flow becomes focused enough to form a stream.

▷ A **source stream** is defined as the stream that reaches from a channel head to a junction with another stream.

▷ Roughly analogous to capillary vessels.

▷ Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.
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Stream Ordering:

1. Label all source streams as order $\omega = 1$ and remove.
2. Label all new source streams as order $\omega = 2$ and remove.
3. Repeat until one stream is left (order $= \Omega$)
4. Basin is said to be of the order of the last stream removed.
5. Example above is a basin of order $\Omega = 3$. 
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Stream Ordering—A large example:

Mississippi

longitude
latitude

ω = 11
10
9
8
Stream Ordering:

Another way to define ordering:

- As before, label all source streams as order $\omega = 1$.
- Follow all labelled streams downstream.
- Whenever two streams of the same order ($\omega$) meet, the resulting stream has order incremented by 1 ($\omega + 1$).

- If streams of different orders $\omega_1$ and $\omega_2$ meet, then the resultant stream has order equal to the largest of the two.

- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where $\delta$ is the Kronecker delta.
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One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ... but relationships based on ordering appear to be robust to resolution changes.
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Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand network architecture
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Resultant definitions:

- A basin of order $\Omega$ has $n_\omega$ streams (or sub-basins) of order $\omega$.
  - $n_\omega > n_{\omega+1}$
- An order $\omega$ basin has area $a_\omega$.
- An order $\omega$ basin has a main stream length $\ell_\omega$.
- An order $\omega$ basin has a stream segment length $s_\omega$.
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Self-similarity of river networks

- First quantified by Horton (1945) \cite{Horton1945}, expanded by Schumm (1956) \cite{Schumm1956}

Three laws:

- Horton’s law of stream numbers:
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  n_\omega / n_{\omega+1} = R_n > 1
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- So... Horton’s laws are defined by three ratios:

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Insert question 2, assignment 2 (田)
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Horton’s laws in the real world:

The Mississippi

The Nile

The Amazon
Horton’s laws-at-large

Blood networks:

- Horton’s laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...
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<th>Symbol</th>
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<tbody>
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▶ Property 1: Scale independence—depends only on difference between orders:

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Tokunaga’s law—an example:

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\[ R_T \approx 4 \]
The Mississippi

A Tokunaga graph:
Nutshell:

Branching networks I:

- Show remarkable **self-similarity** over many scales.
- There are many interrelated scaling laws.
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- **Horton’s laws** reveal self-similarity.
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Unified view of scaling laws for river networks. 

Studies of longitudinal stream profiles in Virginia and Maryland. 

Erosional development of streams and their drainage basins; hydrophysical approach to quantitative morphology. 
References II

*Fractal River Basins: Chance and Self-Organization.*  

Evolution of drainage systems and slopes in badlands at Perth Amboy, New Jersey.  

Hypsometric (area altitude) analysis of erosional topography.  
References III

