1. (9 pts) Consider a family of undirected random networks with degree distribution

\[ P_k = c\delta_{k1} + (1 - c)\delta_{k3} \]

where \( \delta_{ij} \) is the Kronecker delta function where \( c \) is a constant to be determined below. Also allow nodes to be correlated according to the following node-node mixing probability:

\[ E = [e_{ij}] = \begin{bmatrix} e_{00} & e_{02} \\ e_{20} & e_{22} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} (1 + r) & (1 - r) \\ (1 - r) & (1 + r) \end{bmatrix} \]

where \( e_{ij} \) is the probability that a randomly chosen edge connects a node of degree \( i + 1 \) an a node of degree \( j + 1 \), and only the non-zero values of \( E \) are shown.

(a) Determine \( c \) so that purely disassortative networks are achievable if \( r \) is tuned to -1.

(b) Analytically determine the size of the giant component as a function of \( r \).

(c) Determine the size of the largest component containing only degree 3 nodes as a function of \( r \).

Hint: allow degree 3 nodes to be always vulnerable \( (\beta_{3i} = 1 \text{ for } i = 0, 1, 2, \text{ and } 3) \) and degree 1 nodes to be never vulnerable \( (\beta_{1i} = 0 \text{ for } i = 0 \text{ and } 1) \).
2. Spreading on assortative networks: Starting from
\[ \theta_{j,t+1} = G_j(\vec{\theta}_t) = \phi_0 + (1 - \phi_0) \times \]
\[ \sum_{k=1}^{\infty} \frac{e_{j-1,k-1}}{R_{j-1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \theta_{i,k,t} (1 - \theta_{k,t})^{k-1-i} \beta_{ki}. \]

show the matrix for which we must have the largest eigenvalue greater than 1 for spreading to occur is
\[ \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} = \frac{e_{j-1,k-1}}{R_{j-1}} (k - 1)(\beta_{k1} - \beta_{k0}). \]

3. Show that for uncorrelated networks, i.e., when \( e_{jk} = R_j R_k \), the above condition collapses to the standard condition
\[ \sum_{k=1}^{\infty} (k - 1) \frac{k P_k}{\langle k \rangle} (\beta_{k1} - \beta_{k0}) > 1. \]