Complex Networks, CSYS/MATH 303—Assignment 3  
University of Vermont, Spring 2010

**Dispersed:** Saturday, February 13, 2010.  
**Due:** By start of lecture, 10:00 am, Tuesday, February 23, 2010.  

Some useful reminders:

**Instructor:** Peter Dodds  
**Office:** 203 Lord House, 16 Colchester Avenue (TR)  
**E-mail:** peter.dodds@uvm.edu  
**Office hours:** 1:00 pm to 2:30 pm, Wednesday @ Farrell, and by appointment  
**Course website:** [http://www.uvm.edu/~pdodds/teaching/courses/2010-01UVM-303/](http://www.uvm.edu/~pdodds/teaching/courses/2010-01UVM-303/)

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.  

Graduate students are requested to use LaTeX (or related variant).

**Supply networks and allometry:**

1. From lectures on Supply Networks:
   
   Show that for large $V$ and $0 < \epsilon < 1/2$
   
   $$\min V_{\text{net}} \propto \int_{\Omega_{d,D}(V)} \rho ||\vec{x}||^{1-2\epsilon} \, d\vec{x} \sim \rho V^{1+\gamma_{\text{max}}(1-2\epsilon)}$$

   Reminders: we defined $L_i = c_i^{-1} V^{\gamma_i}$ where $\gamma_1 + \gamma_2 + \ldots + \gamma_d = 1$, $\gamma_1 = \gamma_{\text{max}} \geq \gamma_2 \geq \ldots \geq \gamma_d$, and $c = \prod_i c_i \leq 1$ is a shape factor.

   Hints: assume the first $k$ lengths scale in the same way with $\gamma_1 = \ldots = \gamma_k = \gamma_{\text{max}}$, and write $||\vec{x}|| = (x_1^2 + x_2^2 + \ldots + x_d^2)^{1/2}$.

2. Consider a set of rectangular areas with side lengths $L_1$ and $L_2$ such that $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$ where $A$ is area and $\gamma_1 + \gamma_2 = 1$. Assume $\gamma_1 > \gamma_2$ and that $\epsilon = 0$.

   Now imagine that material has to be distributed from a central source in each of these areas to sinks distributed with density $\rho(A)$, and that these sinks draw the same amount of material per unit time independent of $L_1$ and $L_2$.

   Find an exact form for how the volume of the most efficient distribution network scales with overall area $A = L_1 L_2$. (Hint: you will have to set up a double integration over the rectangle.)

   If network volume must remain a constant fraction of overall area, determine the maximal scaling of sink density $\rho$ with $A$.  

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3. (a) For a family of \(d\)-dimensional regions, with scaling as per Question 1, determine, to leading order, the scaling of hyper-surface area \(S\) with volume \(V\). In other words, find the exponent \(\beta\) in \(S \propto V^\beta\) as \(V \to \infty\).

Assume that nothing peculiar happens with the shapes (as we have always implicitly done), in that there is no fractal roughening.

Hint: figure out how the circumference for the rectangles in the previous question scales with area \(A\). For \(d\) dimensions, think about how the hyper-surface area of a hyperrectangle (or orthotope) would scale.

(b) For \(d = 3\), what is the maximum possible value of \(\beta\) and for what values of the \(\gamma_i\) does this occur?