A COMPLETE THERMODYNAMIC ANALOGY FOR LANDSCAPE EVOLUTION (*)

A.E. SCHEIDEGGER (‡)

ABSTRACT

The analogy embodying an entropy concept for landscape evolution can be extended to all other thermodynamic functions.

INTRODUCTION

Leopold and Langbein (1962) recently postulated an analogy between landscape evolution and the nonsteady-state temperature distribution in a planar medium. Their argument is based on the concept of entropy and thus on a formal analogy with regard to the second principle of thermodynamics, between landscape, and temperature fields. Scheidegger (1964) has shown that there is a statistical justification for this, originally purely formal, analogy.

Since the success in explaining the evolution of landforms by means of the entropy-analogy is profound, the question arose as to whether the analogy with thermodynamics could not be extended further than originally envisaged. As has been noted above, the analogy, up to this point, pertains only to the entropy concept, i.e. it involves only the second principle of thermodynamics. One might expect that there also ought to be phenomena in landscape evolution that would be governed by a corollary of the first principle of thermodynamics. In other words, it might be expected that there is a complete analogy between landscape evolution and the (two-dimensional) nonsteady state temperature distribution in an ideal gas.

It is the aim of this paper to investigate the possibility of such a complete temperature analogy, and to show that the latter, indeed, exists.

THE COMPLETE CORRESPONDENCES

In order to fix the background of our investigation, we recall the analogy relations of Leopold and Langbein (1962) between a temperature field and a landscape.

The temperature field is described by the temperature $T$; the quantity of heat $Q$ is associated with a temperature. The planar Cartesian coordinates are $x$ and $y$.

The landscape is described by the elevation $h$ of a point above sea level; the mass $M$ is associated with an elevation. The planar Cartesian coordinates are again $x$ and $y$.

The analogy between a thermal field and a landscape then maintains the following correspondences:

$$ T \rightarrow h $$

$$ dQ \rightarrow dM $$

Based on the above, it is possible to define corresponding entropies ($dS = dQ/T \rightarrow dM/h$) and other thermodynamic properties. Furthermore, the quantity of heat introduced in a given substance is given by

$$ dQ = \gamma dT $$

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with $\gamma$ being a heat capacity coefficient. The analog of this in a landscape is
\[ dM = \gamma dh \]
where $\gamma$ is now an analog of the heat-capacity coefficient.

Our task is now to extend the above correspondences to energy terms. For a regular thermodynamic system, the first principle of thermodynamics states (see e.g. Planck, 1945):
\[ U_2 - U_1 = Q + W \]
or, in differentials
\[ dU = dQ + dW \]
where $U$ is the internal energy, $Q$ is the quantity of heat introduced from outside and $W$ the work performed externally on the system. In landscape evolution, one would like to have, therefore, a similar relation, viz.
\[ \eta_2 - \eta_1 = M + W \]
or, in differentials
\[ d\eta = dM + dW \]
where $\eta$ now signifies some potential, $M$ the mass that was introduced and $W$ some "fictitious work" whose physical meaning has yet to be defined.

For an ideal gas, $W$ is
\[ W = -\int p dV \]
Here, $V$ is the geometric domain in which the variables vary, and $p$ the pressure. Because of the ideal gas law, the latter can be expressed as follows
\[ p = \frac{RT}{V} = \text{const.} \frac{T}{V} \]

The last relation yields a means of setting up an analogy to "pressure" in landscapes. In the latter, $V$ corresponds to the area $A$ under consideration, $T$ is the height $h$ (see above) so that one has
\[ p_{\text{landscape}} = \text{const.} \frac{h}{A} \]
at least in the equilibrium case.

If we are essentially interested in an "average" geographic cross section across a landscape, we have only one space coordinate ($x$); denoting the total length of the section by $L$, we have (denoting the constant by $x$)
\[ p_{\text{landscape}} = \text{const.} \frac{h}{L} = \frac{x}{L} \]
The analog of work is then
\[ W = -\int p dV = -\int \frac{x(h/L)}{L} dL \]
The equilibrium case, which is here under discussion, would therefore be illustrated, so to say, by a "box of sand", or by an alluvial fan with a straight surface-section (fig. 1) and $h$ denoting the average height.

The relationships for the "pressure" in a landscape and the potential already establish the complete analogy between thermodynamics and certain variables in landscape theory.

\[ Q_2 - Q_1 = W_2 - W_1 + W_3 - W_4, \]

The second principle implies
\[ \frac{Q_2}{T_2} - \frac{Q_1}{T_1} = 0. \]

To set up the analogy of the above process in a landscape, we assume that a landscape section composed of a certain mass of rock of length $L$ with the average height $h$ at surface above some given base level. The steps of the "Carnot" cycle (see fig. 2) are then:

1. The landscape section is extended from $L_1$ to $L_2$, with $h$ being held at $h_1$. In order to do this, the mass $M_1$ must be added to the landscape, and the value of $W$ is
\[ W_1 = \int_{L_1}^{L_2} \alpha(h_1/L) \, dL = \alpha h_1 \log \frac{L_2}{L_1}. \]
2. The landscape material is compressed until the average height reaches $h_2$. No mass is added or subtracted. The quantity $W_2$ is

$$W_2 = \int_{h = h_1}^{h = h_2} \frac{\beta}{\beta} \frac{h}{\beta} dL(h)$$

and since mass is constant, $hL = \text{const} = \beta$. The quantity $W_2$ follows:

$$W_2 = \int_{h_1}^{h_2} \frac{h^2}{\beta} \frac{\beta}{h^2} = dh = \int_{h_1}^{h_2} x dh = \alpha(h_2 - h_1)$$

or

$$W_2 = \alpha(h_2 - h_1)$$

3. The landscape material is "compressed" some more at constant average height $h_2$. An amount $M_2$ of mass is taken from the system while $W_3$ is

$$W_3 = -\int_{L_1}^{L_2} \frac{h_2}{L} dL = \beta h_2 \log \frac{L_2}{L_1}$$

4. An expansion occurs with no mass added or subtracted until $h$ drops from $h_2$ to $h_1$. We have

$$W_4 = \alpha(h_2 - h_1)$$

Note that all quantities $W$ are defined so as to be positive. The cycle is now closed and the model is in its original state. The first principle of thermodynamics (i.e. its analog in the present case) states

$$M_2 - M_1 = W_2 - W_1 + W_3 - W_4$$

$$= \alpha(h_2 - h_1) - \beta h_1 \log \frac{L_2}{L_1} + \beta h_2 \log \frac{L_2}{L_4} - \alpha(h_2 - h_1)$$

$$= \alpha \left[ -h_1 \log \frac{L_2}{L_1} + h_2 \log \frac{L_2}{L_4} \right]$$

note that $L_A > L_1$, $L_2 > L_1$, $h_2 > h_1$. Thus

$$M_2 - M_1 = \alpha \left[ h_2 \log \frac{L_2}{L_4} - h_1 \log \frac{L_2}{L_1} \right]$$

The last is a relation valid for a Carnot process of the type envisaged.

Next, the second fundamental principle of thermodynamics (i.e. its analogy in landscape evolution) states

$$\frac{M_2}{h_2} \frac{M_1}{h_1} = \alpha$$
The latter is nothing but the expression of the fact that an addition of material to a landscape will make its elevation proportionately higher, so that the last equation can be taken as an expression of the law of conservation of mass. It is clear that this must be so, since the analogy of entropy in landscape evolution is justifiable (as shown by Scheidegger, 1964) by assuming that the process is a statistical one with mass being conserved.

Thus, the second principle leads to a further confirmation of the analogy postulated by Leopold and Langbein (1962) and verifies the contention of Scheidegger (1964) that this analogy is possible because mass must be conserved.

It is possible to illustrate the Carnot cycle, for instance, with a hypothetical “alluvial fan” (see fig. 2). The height \( h \) refers to the mid-point of the fan (“average height”). Of course, step 2 is not one that is likely to occur in nature without external action, but neither is the corresponding step in gas-thermodynamics!

Since, in thermodynamics the Carnot process is one transforming work into heat (or vice versa), in our analogy it connects the variables \( W \) and \( M \).

**THERMODYNAMIC POTENTIALS**

With the definition of \( W \), it is now possible to set up a complete analog to thermodynamic functions in landscape theory.

The “potential” \( U \) is defined by

\[
dU = dM + dW
\]

Since we have been able to give a meaning to the quantity \( W \) in landscapes, it is clear that one now is able to assign a meaning to the potential \( U \). The analog of entropy, \( S \), was already defined earlier by Leopold and Langbein; it is clear that all ordinary thermodynamic functions have now found an equivalent in landscape theory. To recapitulate, we have (with \( x \) and \( y \) being...
\[
E \left[ \sum_{i=1}^{N} x_i \right] = \sum_{i=1}^{N} E[x]
\]

(4)

\[
\left[ \sum_{i=1}^{N} x_i \right] \sim \mathcal{N} \left( \sum_{i=1}^{N} E[x], \sum_{i=1}^{N} Var[x] \right)
\]

(5)

Let \( \bar{x} \) be the mean of the dependent samples. The influence of the mean of the \( N \) samples can be divided by expressing the variance in terms of the coefficients given in equation (8), sum up \( g \) over \( g \) and \( g \) by the coefficients given in equation (9).

\[
\sum_{i=1}^{N} Var[x] = \sum_{i=1}^{N} \left[ \sum_{i=1}^{N} Var[x] \right]
\]

(6)

\[
\left[ \sum_{i=1}^{N} x_i \right] \sim \mathcal{N} \left( \sum_{i=1}^{N} E[x], \sum_{i=1}^{N} Var[x] \right)
\]

(7)

With a loss in general, assume \( E = 0 \) and

\[
\left[ (y) \right] = E \left[ (x) \right] = 0
\]

(8)

Then, for a given second moment process, this is not always true, the variance under certain conditions is given in equation (10). A simple model of the dependent samples will illustrate this point. Dependent samples will contribute information in a dependent manner. The dependent variance will contribute more information than independent cases. In some cases, it will contribute more information than the independent cases. This order process can be derived to the second order in Taylor's expansion. Where the second order term is the variance in which make up the second order term.

\[
\left[ (y) \right] = E \left[ (x) \right] = 0
\]

(9)

\[
\left[ (x) \right] = E \left[ (x) \right] = 0
\]

(10)

\[
\left[ (x) \right] = E \left[ (x) \right] = 0
\]

(11)

where \( \sigma \) is the variance of both the dependent samples.
the variance of both the dependent and independent series. The summation of the
cross products can be written
\[ \sum_{i=1}^{N} \sum_{j=i+1}^{N} x_i x_j = x_1 x_2 + x_1 x_3 + \ldots + x_1 x_{N-1} + x_1 x_N + x_2 x_3 + x_2 x_4 + \ldots + x_2 x_N + \ldots + x_{N-1} x_N \]

Hence
\[ E(\sum_{i=1}^{N} \sum_{j=i+1}^{N} x_i x_j) = \sigma^2 \left( \sum_{k=1}^{N-1} \rho_k + \sum_{k=2}^{N} \rho_k + \ldots + \sum_{k=N-2}^{N} \rho_k \right) \tag{8} \]

where \( \rho_k \) is the \( k \)th autocorrelation. For the second order Markov process \( \varrho \) can be written as (Kendall: 1951)
\[ \rho_k = \frac{c^k \sin(k\theta + \psi)}{\sin\psi} \tag{9} \]
where
\[ c = \sqrt{-b} \]
\[ \cos \theta = \frac{a}{2\sqrt{-b}} \]
\[ \tan \psi = \frac{1 + c^2}{1 - c^2} \tan \theta \tag{10} \]

\[ \rho_k = \frac{\tan \psi}{1 - 2c \cos \theta + c^2} \sum_{i=2}^{N} x_i x_j \]

Sum up \( \varrho \) over \( \sin k\theta \) and \( \cos k\theta \) (Jolley: 1961), thus giving compact expressions for each summation in equation (8),
\[ E(\Sigma x_i x_j) = \sigma^2 (A + B) \tag{12} \]

where
\[ A = \frac{\cot \psi}{1 - 2c \cos \theta + c^2} \left[ Nc \sin \theta - \left[ 2c^{N-1} \sin N\theta - c^{N+1} \sin(N+1)\theta - c^{N+3} \sin(N-1)\theta + c(1-c^2) \sin \theta \right] \left[ 1 - 2c \cos \theta + c^2 \right]^{-1} \right] \]
\[ B = \frac{1}{(1 - 2c \cos \theta + c^2)^2} \left[ c^{N+3} \cos(N-1)\theta - 2c^{N+2} \cos N\theta + c^{N+1} \cos(N+1)\theta - c^3 \cos \theta + Nc \cos \theta - Nc^3 - 2Nc^2 \cos^2 \theta - Nc^4 + 3Nc^3 \cos \theta - c \cos \theta + 2c \right] \tag{13} \]
Hence

$$\text{Var}(\bar{x}) = \frac{1}{N^2} [N\sigma^2 + 2\sigma^2(A + B)]$$  \hspace{1cm} (14)$$

The effective number of observations $N'$ is the number of random events whose variance of the mean equals the variance of the mean for a sequence of autocorrelated events.

The variance of the mean $\bar{x}'$ for the $N'$ random events is

$$\text{Var}(\bar{x}') = \frac{\sigma^2}{N'}$$  \hspace{1cm} (15)$$

Equating (14) to (15) and solving for $N'$,

$$N' = N \left[ 1 + \frac{2}{N}(A + B) \right]^{-1}$$  \hspace{1cm} (16)$$

Tables 1-7 give values of $N'$ for $N = 5, 10, 20, 30, 40, 50, 100$ and for $0 \leq \rho_1 \leq 0.9$ and $-0.9 \leq \rho_2 \leq 0.9$. No negative values of $\rho_2$ are considered because hydrologic phenomena yield positive first order autocorrelations. The asterisks in the table represent certain values of $\rho_1$ and $\rho_2$ which are inadmissible due to mathematical constraints on the second order Markov process. These constraints are (Kendall: 1951),

$$-1 < \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} < 0$$

$$\left[ \frac{\rho_1 - \rho_1 \rho_2}{1 - \rho_1^2} \right]^2 < 4 \left| \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \right|$$

$$\left| \frac{\rho_1 - \rho_1 \rho_2}{1 - \rho_1^2} \right| / 2 < 1$$

$$\sqrt{-\frac{(\rho_2 - \rho_1^2)}{(1 - \rho_1^2)} - \frac{(\rho_1 - \rho_1 \rho_2)^2}{4 - \rho_2^2}} < 1$$  \hspace{1cm} (17)$$

It is evident from tables 1-7 that negative values of $\rho_2$ yield high values of $N'$ such that $N' > N$. Only for $\rho_2 = -0.1$ are there some values of $N'$ such that $N' \leq N$. In many cases $N'$ is much greater than $N$, and as $\rho_2$ increases, so does $N'$. For example, when $N = 30, \rho_1 = 0.3, \rho_2 = -0.1$, then $N' = 850$. That is, 30 dependent events are contributing as much information as 850 random events—a startling result indeed! This is an extreme situation, however. For a more realistic example let $N = 30, \rho_1 = 0.3, \rho_2 = -0.2$. Then $N' = 31$. One can speculate as to what is happening to produce such high values of $N'$, and it would appear that the negative serial correlations are responsible.

As an example of the effect negative serial correlation has on a system, consider the case where $\rho_2 = \rho_1^2$, so that the Markov process is first order. The effective sample size $N'$ for the process (Dawdy and Matalas: 1964) is

$$N' = N \left[ 1 + \frac{2}{N} \left( \frac{N\rho_1(1-\rho_1) - \rho_1(1-\rho_1^2)}{(1-\rho_1)^2} \right) \right]^{-1}$$  \hspace{1cm} (18)$$
When \( N = 30, q_1 = 0.3 \), then \( N' = 24 \). But when \( N = 30 \) and \( q_1 = -0.3 \), then \( N' = 54 \). The negative correlation is adding more information than the positive serial correlation is taking away.

**SUMMARY AND CONCLUSIONS**

From the values of \( N' \) in tables 1-7, it would seem that for the second order Markov process negative values of \( q_1 \) are working more for the investigator than the positive values of \( q_1 \) and are working against him. That is, sequences generated by the second order Markov process provide more information than sequences generated by the first order Markov process. Whether the second order process could accommodate certain hydrologic time series, as well as meteorologic geochronologic sequences, requires further research.

**REFERENCES**


