Scaling—a Plenitude of Power Laws
Principles of Complex Systems
Course CSYS/MATH 300, Fall, 2009

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Definitions

General observation:
Systems (complex or not) that cross many spatial and temporal scales often exhibit some form of scaling.
**Definitions**

A **power law** relates two variables $x$ and $y$ as follows:

$$ y = cx^\alpha $$

- $\alpha$ is the **scaling exponent** (or just exponent)
- ($\alpha$ can be any number in principle but we will find various restrictions.)
- $c$ is the **prefactor** (which can be important!)

**Looking at data**

- Power-law relationships are linear in log-log space:
  $$ y = cx^\alpha $$
  $$ \Rightarrow \log_b y = \alpha \log_b x + \log_b c $$
  with slope equal to $\alpha$, the scaling exponent.
- Much searching for straight lines on log-log or double-logarithmic plots.
- Good practice: Always, always, always use base 10.
- Talk only about orders of magnitude (powers of 10).

**A beautiful, heart-warming example:**

$$ \alpha \simeq 1.23 $$

- gray matter: ‘computing elements’
- white matter: ‘wiring’

from Zhang & Sejnowski, PNAS (2000) [26]
Why is $\alpha \approx 1.23$?

Quantities (following Zhang and Sejnowski):
- $G = $ Volume of gray matter (cortex/processors)
- $W = $ Volume of white matter (wiring)
- $T = $ Cortical thickness (wiring)
- $S = $ Cortical surface area
- $L = $ Average length of white matter fibers
- $p = $ density of axons on white matter/cortex interface

A rough understanding:
- $G \sim ST$ (convolutions are okay)
- $W \sim \frac{1}{2} pSL$
- $G \sim L^3$ ← this is a little sketchy...
- Eliminate $S$ and $L$ to find $W \propto G^{4/3} / T$

Trickiness:
- With $V = G + W$, some power laws must be approximations.
- Measuring exponents is a hairy business...

A rough understanding:
- We are here: $W \propto G^{4/3} / T$
- Observe weak scaling $T \propto G^{0.10 \pm 0.02}$.
- (Implies $S \propto G^{0.9} \rightarrow$ convolutions fill space.)
- $\Rightarrow W \propto G^{4/3} / T \propto G^{1.23 \pm 0.02}$

Good scaling:

General rules of thumb:
- **High quality**: scaling persists over three or more orders of magnitude for each variable.
- **Medium quality**: scaling persists over three or more orders of magnitude for only one variable and at least one for the other.
- **Very dubious**: scaling ‘persists’ over less than an order of magnitude for both variables.
Unconvincing scaling:

Average walking speed as a function of city population:

Two problems:
1. use of natural log, and
2. minute variation in dependent variable.

from Bettencourt et al. (2007) [3]; otherwise very interesting!

Definitions

Power laws are the signature of scale invariance:

Scale invariant 'objects' look the 'same' when they are appropriately rescaled.

- Objects = geometric shapes, time series, functions, relationships, distributions,...
- 'Same' might be 'statistically the same'
- To rescale means to change the units of measurement for the relevant variables

Scale invariance

Our friend $y = cx^\alpha$:
- If we rescale $x$ as $x = rx'$ and $y$ as $y = r^\alpha y'$,
- then
  $r^\alpha y' = c(rx')^\alpha$
- $y' = cr^\alpha x'^\alpha r^{-\alpha}$
- $y' = cx'^{\alpha}$

Compare with $y = ce^{-\lambda x}$:
- If we rescale $x$ as $x = rx'$, then
  $y = ce^{-\lambda rx'}$
- Original form cannot be recovered.
- $\Rightarrow$ scale matters for the exponential.
Scale invariance

More on $y = ce^{-\lambda x}$:

- Say $x_0 = 1/\lambda$ is the characteristic scale.
- For $x \gg x_0$, $y$ is small, while for $x \ll x_0$, $y$ is large.
- $\Rightarrow$ More on this later with size distributions.

Definitions:

Isometry:
dimensions scale linearly with each other.

Allometry:
dimensions scale nonlinearly.

Allometry (♃):
[refers to] differential growth rates of the parts of a living organism's body part or process.

Confusingly, we use allometric scaling to refer to both:
1. nonlinear scaling (e.g., $x \propto y^{1/3}$)
2. and the relative scaling of different measures (e.g., resting heart rate as a function of body size)
A wonderful treatise on scaling:

McMahon and Bonner, 1983 [18]

The many scales of life:

For the following slide:

Medium-sized creatures (above). 1, Dog; 2, common herring; 3, the largest egg (Aepyornis); 4, song thrush with egg; 5, the smallest bird (hummingbird) with egg; 6, queen bee; 7, common cockroach; 8, the largest stick insect; 9, the largest polyp (Branchiocerianthus); 10, the smallest mammal (flying shrew); 11, the smallest vertebrate (a tropical frog); 12, the largest frog (goliath frog); 13, common grass frog; 14, house mouse; 15, the largest land snail (Achatina) with egg; 16, common snail; 17, the largest beetle (goliath beetle); 18, human hand; 19, the largest starfish (Luidia); 20, the largest free-moving protozoan (an extinct nummulite).
The many scales of life:

Small, "naked-eye" creatures (lower left). 1. One of the smallest fishes (Trimmatom nanus); 2, common brown hydra, expanded; 3, housefly; 4, medium-sized ant; 5, the smallest vertebrate (a tropical frog, the same as the one numbered 11 in the figure above); 6, flea (Xenopsylla cheopis); 7, the smallest land snail; 8, common water flea (Daphnia).

The smallest "naked-eye" creatures and some large microscopic animals and cells (below right). 1, Vorticella, a ciliate; 2, the largest ciliate protozoan (Bursaria); 3, the smallest many-celled animal (a rotifer); 4, smallest flying insect (Eiaphis); 5, another ciliate (Paramecium); 6, cheese mite; 7, human sperm; 8, human ovum; 9, dysentery ameoba; 10, human liver cell; 11, the foreleg of the flea (numbered 6 in the figure to the left).
Non-uniform growth:

Scaling-at-large
Allometry
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Examples
History: Metabolism
Measuring exponents
History: River networks
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Frame 31/117

Non-uniform growth—arm length versus height:

Good example of a break in scaling:

A crossover in scaling occurs around a height of 1 metre.

p. 32, McMahon and Bonner

Frame 32/117

Weightlifting: $M_{\text{worldrecord}} \propto M_{\text{lifter}}^{2/3}$

Idea: Power $\sim$ cross-sectional area of isometric lifters.

p. 53, McMahon and Bonner

Frame 33/117

Titanothere horns: $L_{\text{horn}} \sim L_{\text{skull}}^4$

p. 36, McMahon and Bonner

Frame 34/117
The allometry of nails:

- Diameter $\propto \text{Mass}^{3/8}$
- Length $\propto \text{Mass}^{1/4}$
- Diameter $\propto \text{Length}^{2/3}$

p. 58–59, McMahon and Bonner [18]

A buckling instability?:

- Physics/Engineering result: Columns buckle under a load which depends on $d^4/\ell^2$.
- To drive nails in, resistive force $\propto$ nail circumference $= \pi d$.
- Match forces independent of nail size: $d^4/\ell^2 \propto d$.
- Leads to $d \propto \ell^{2/3}$.

Rowing: Speed $\propto$ (number of rowers)$^{1/9}$

Scaling in Cities:

- “Growth, innovation, scaling, and the pace of life in cities”
  Bettencourt et al., PNAS, 2007. [3]
- Quantified levels of
  - Infrastructure
  - Wealth
  - Crime levels
  - Disease
  - Energy consumption as a function of city size $N$ (population).
Scaling in Cities:

Table 1. Scaling exponents for urban indicators vs. city size

<table>
<thead>
<tr>
<th>Indicator</th>
<th>( \gamma )</th>
<th>( \beta )</th>
<th>95% CI</th>
<th>Adj.( R^2 )</th>
<th>Observations</th>
<th>Country-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>New patents</td>
<td>1.27</td>
<td>0.72</td>
<td>[1.25,1.29]</td>
<td>0.72</td>
<td>331</td>
<td>U.S. 2001</td>
</tr>
<tr>
<td>Inventors</td>
<td>1.25</td>
<td>0.76</td>
<td>[1.22,1.27]</td>
<td>0.76</td>
<td>331</td>
<td>U.S. 2001</td>
</tr>
<tr>
<td>Private R&amp;D employment</td>
<td>1.34</td>
<td>0.92</td>
<td>[1.29,1.39]</td>
<td>0.92</td>
<td>266</td>
<td>U.S. 2002</td>
</tr>
<tr>
<td>“Supercreative” employment</td>
<td>1.15</td>
<td>0.89</td>
<td>[1.11,1.18]</td>
<td>0.89</td>
<td>287</td>
<td>U.S. 2003</td>
</tr>
<tr>
<td>R&amp;D establishments</td>
<td>1.19</td>
<td>0.77</td>
<td>[1.14,1.22]</td>
<td>0.77</td>
<td>287</td>
<td>U.S. 1997</td>
</tr>
<tr>
<td>R&amp;D employment</td>
<td>1.26</td>
<td>0.93</td>
<td>[1.18,1.43]</td>
<td>0.93</td>
<td>295</td>
<td>China 2002</td>
</tr>
<tr>
<td>Total wages</td>
<td>1.12</td>
<td>0.96</td>
<td>[1.09,1.13]</td>
<td>0.96</td>
<td>361</td>
<td>U.S. 2002</td>
</tr>
<tr>
<td>Total bank deposits</td>
<td>1.08</td>
<td>0.91</td>
<td>[1.03,1.11]</td>
<td>0.91</td>
<td>267</td>
<td>U.S. 1996</td>
</tr>
<tr>
<td>GDP</td>
<td>1.15</td>
<td>0.96</td>
<td>[1.06,1.23]</td>
<td>0.96</td>
<td>295</td>
<td>China 2002</td>
</tr>
<tr>
<td>GDP</td>
<td>1.26</td>
<td>0.64</td>
<td>[1.09,1.46]</td>
<td>0.64</td>
<td>196</td>
<td>EU 1999-2003</td>
</tr>
<tr>
<td>Total electrical consumption</td>
<td>1.07</td>
<td>0.88</td>
<td>[1.03,1.11]</td>
<td>0.88</td>
<td>392</td>
<td>Germany 2002</td>
</tr>
<tr>
<td>New AIDS cases</td>
<td>1.23</td>
<td>0.76</td>
<td>[1.18,1.29]</td>
<td>0.76</td>
<td>93</td>
<td>U.S. 2002-2003</td>
</tr>
<tr>
<td>Serious crimes</td>
<td>1.16</td>
<td>0.89</td>
<td>[1.11,1.18]</td>
<td>0.89</td>
<td>287</td>
<td>U.S. 2003</td>
</tr>
<tr>
<td>Total housing</td>
<td>1.00</td>
<td>0.99</td>
<td>[0.99,1.01]</td>
<td>0.99</td>
<td>316</td>
<td>U.S. 1990</td>
</tr>
<tr>
<td>Total employment</td>
<td>1.01</td>
<td>0.98</td>
<td>[0.99,1.02]</td>
<td>0.98</td>
<td>331</td>
<td>U.S. 2001</td>
</tr>
<tr>
<td>Household electrical coverage</td>
<td>1.00</td>
<td>0.88</td>
<td>[0.94,1.06]</td>
<td>0.88</td>
<td>377</td>
<td>Germany 2002</td>
</tr>
<tr>
<td>Household water consumption</td>
<td>1.01</td>
<td>0.96</td>
<td>[0.89,1.11]</td>
<td>0.96</td>
<td>295</td>
<td>China 2002</td>
</tr>
<tr>
<td>Gasoline stations</td>
<td>0.77</td>
<td>0.93</td>
<td>[0.74,0.81]</td>
<td>0.93</td>
<td>318</td>
<td>U.S. 2001</td>
</tr>
<tr>
<td>Gasoline sales</td>
<td>0.79</td>
<td>0.94</td>
<td>[0.73,0.80]</td>
<td>0.94</td>
<td>318</td>
<td>U.S. 2001</td>
</tr>
<tr>
<td>Length of electrical cables</td>
<td>0.87</td>
<td>0.75</td>
<td>[0.82,0.92]</td>
<td>0.75</td>
<td>380</td>
<td>Germany 2002</td>
</tr>
<tr>
<td>Road surface</td>
<td>0.83</td>
<td>0.87</td>
<td>[0.74,0.92]</td>
<td>0.87</td>
<td>29</td>
<td>Germany 2002</td>
</tr>
</tbody>
</table>

Data sources are shown in SI Text. CI, confidence interval; Adj.\( R^2 \), adjusted \( R^2 \); GDP, gross domestic product.

Intriguing findings:
- Global supply costs scale sublinearly with \( N (\beta < 1) \).
- Returns to scale for infrastructure.
- Total individual costs scale linearly with \( N (\beta = 1) \).
- Individuals consume similar amounts independent of city size.
- Social quantities scale superlinearly with \( N (\beta > 1) \).
- Creativity (# patents), wealth, disease, crime, ...

Density doesn’t seem to matter...
- Surprising given that across the world, we observe two orders of magnitude variation in area covered by agglomerations of fixed populations.

A focus:
- How much energy do organisms need to live?
- And how does this scale with organismal size?

Ecology—Species-area law: \( N_{\text{species}} \propto A^{\beta} \)

Allegedly (data is messy):
- On islands: \( \beta \approx 1/4. \)
- On continuous land: \( \beta \approx 1/8. \)
Animal power

Fundamental biological and ecological constraint:

\[ P = c M^\alpha \]

\( P \) = basal metabolic rate  
\( M \) = organismal body mass

What one might expect:

\( \alpha = 2/3 \) because . . .

- Dimensional analysis suggests an energy balance surface law:
  \[ P \propto S \propto V^{2/3} \propto M^{2/3} \]

- Lognormal fluctuations:  
  Gaussian fluctuations in \( \log P \) around \( \log cM^\alpha \).

- Stefan-Boltzmann relation for radiated energy:
  \[ \frac{dE}{dt} = \sigma \varepsilon ST^4 \]

The prevailing belief of the church of quarterology

\( \alpha = 3/4 \)

\[ P \propto M^{3/4} \]

Huh?

Prefactor \( c \) depends on body plan and body temperature:

- Birds 39–41 °C
- Eutherian Mammals 36–38 °C
- Marsupials 34–36 °C
- Monotremes 30–31 °C
Related putative scalings:

- number of capillaries $\propto M^{3/4}$
- time to reproductive maturity $\propto M^{1/4}$
- heart rate $\propto M^{-1/4}$
- cross-sectional area of aorta $\propto M^{3/4}$
- population density $\propto M^{-3/4}$

The great ‘law’ of heartbeats:

Assuming:

- Average lifespan $\propto M^{\beta}$
- Average heart rate $\propto M^{-\beta}$
- Irrelevant but perhaps $\beta = 1/4$.

Then:

- Average number of heart beats in a lifespan $\propto (\text{Average lifespan}) \times (\text{Average heart rate}) = M^{\beta - \beta} = M^0$
- Number of heartbeats per life time is independent of organism size!
- $\approx 1.5$ billion....

History

1840’s: Sarrus and Rameaux \cite{22} first suggested $\alpha = 2/3$.

1883: Rubner \cite{21} found $\alpha \simeq 2/3$. 
1930's: Brody, Benedict study mammals.\[6\]
Found $\alpha \simeq 0.73$ (standard).

1932: Kleiber analyzed 13 mammals.\[15\]
Found $\alpha = 0.76$ and suggested $\alpha = 3/4$.

1950/1960: Hemmingsen\[12, 13\]
Extension to unicellular organisms.
$\alpha = 3/4$ assumed true.

1964: Troon, Scotland: \[4\]
3rd symposium on energy metabolism.
$\alpha = 3/4$ made official . . .
\[29\] to zip.
3/4 is held by many to be the one true exponent.

In the Beat of a Heart: Life, Energy, and the Unity of Nature—by John Whitfield

But—much controversy...

See ‘Re-examination of the “3/4-law” of metabolism’ Dodds, Rothman, and Weitz [9]

Heusner’s data (1991) [14]
391 Mammals
blue line: 2/3
red line: 3/4.
(B = P)

Bennett and Harvey’s data (1987) [2]
398 birds
blue line: 2/3
red line: 3/4.
(B = P)

Important:

Ordinary Least Squares (OLS) Linear regression is only appropriate for analyzing a dataset \{ (x_i, y_i) \} when we know the x_i are measured without error.

Here we assume that measurements of mass M have less error than measurements of metabolic rate B.

Linear regression assumes Gaussian errors.
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Measuring exponents

More on regression:
If (a) we don’t know what the errors of either variable are, or (b) no variable can be considered independent, then we need to use Standardized Major Axis Linear Regression. (aka Reduced Major Axis = RMA.)

For Standardized Major Axis Linear Regression:

\[
\text{slope}_{\text{SMA}} = \frac{\text{standard deviation of } y \text{ data}}{\text{standard deviation of } x \text{ data}}
\]

Very simple!

Heusner’s data, 1991 (391 Mammals)

<table>
<thead>
<tr>
<th>range of M</th>
<th>N</th>
<th>(\hat{\alpha})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\leq 0.1 \text{ kg})</td>
<td>167</td>
<td>0.678 (\pm) 0.038</td>
</tr>
<tr>
<td>(\leq 1 \text{ kg})</td>
<td>276</td>
<td>0.662 (\pm) 0.032</td>
</tr>
<tr>
<td>(\leq 10 \text{ kg})</td>
<td>357</td>
<td>0.668 (\pm) 0.019</td>
</tr>
<tr>
<td>(\leq 25 \text{ kg})</td>
<td>366</td>
<td>0.669 (\pm) 0.018</td>
</tr>
<tr>
<td>(\leq 35 \text{ kg})</td>
<td>371</td>
<td>0.675 (\pm) 0.018</td>
</tr>
<tr>
<td>(\leq 350 \text{ kg})</td>
<td>389</td>
<td>0.706 (\pm) 0.016</td>
</tr>
<tr>
<td>(\leq 3670 \text{ kg})</td>
<td>391</td>
<td>0.710 (\pm) 0.021</td>
</tr>
</tbody>
</table>

Measuring exponents

Relationship to ordinary least squares regression is simple:

\[
\text{slope}_{\text{SMA}} = r^{-1} \times \text{slope}_{\text{OLS } y \text{ on } x} = r \times \text{slope}_{\text{OLS } x \text{ on } y}
\]

where \(r\) = standard correlation coefficient:

\[
r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}
\]
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Hypothesis testing

Test to see if \( \alpha' \) is consistent with our data \( \{(M_i, B_i)\} \):

\[
H_0 : \alpha = \alpha' \quad \text{and} \quad H_1 : \alpha \neq \alpha'.
\]

Assume each \( B_i \) (now a random variable) is normally distributed about \( \alpha' \log_{10} M_i + \log_{10} c \).

Follows that the measured \( \alpha \) for one realization obeys a \( t \) distribution with \( N - 2 \) degrees of freedom.

Calculate a \( p \)-value: probability that the measured \( \alpha \) is as least as different to our hypothesized \( \alpha' \) as we observe.

(see, for example, DeGroot and Scherish, “Probability and Statistics”[^7])

Revisiting the past—mammals

<table>
<thead>
<tr>
<th>Full mass range:</th>
<th>( M_{\text{max}} )</th>
<th>( N )</th>
<th>( \hat{\alpha} )</th>
<th>( p_{2/3} )</th>
<th>( p_{3/4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M ( \leq 10 ) kg:</td>
<td>( 13 )</td>
<td>0.738</td>
<td>( &lt; 10^{-6} )</td>
<td>0.11</td>
<td></td>
</tr>
<tr>
<td>Brody</td>
<td>( 35 )</td>
<td>0.718</td>
<td>( &lt; 10^{-4} )</td>
<td>10^{-2}</td>
<td></td>
</tr>
<tr>
<td>Heusner</td>
<td>( 391 )</td>
<td>0.710</td>
<td>( &lt; 10^{-6} )</td>
<td>10^{-5}</td>
<td></td>
</tr>
<tr>
<td>Bennett and Harvey</td>
<td>398</td>
<td>0.664</td>
<td>0.69</td>
<td>( &lt; 10^{-15} )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( M \geq 10 ) kg:</th>
<th>( N )</th>
<th>( \hat{\alpha} )</th>
<th>( p_{2/3} )</th>
<th>( p_{3/4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M ( \leq 10 ) kg:</td>
<td>( 5 )</td>
<td>0.667</td>
<td>0.99</td>
<td>0.088</td>
</tr>
<tr>
<td>Brody</td>
<td>( 26 )</td>
<td>0.709</td>
<td>( &lt; 10^{-3} )</td>
<td>( &lt; 10^{-3} )</td>
</tr>
<tr>
<td>Heusner</td>
<td>( 357 )</td>
<td>0.668</td>
<td>0.91</td>
<td>( &lt; 10^{-15} )</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>( M \geq 10 ) kg:</th>
<th>( N )</th>
<th>( \hat{\alpha} )</th>
<th>( p_{2/3} )</th>
<th>( p_{3/4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M ( \leq 10 ) kg:</td>
<td>( 8 )</td>
<td>0.754</td>
<td>( &lt; 10^{-4} )</td>
<td>0.66</td>
</tr>
<tr>
<td>Brody</td>
<td>( 9 )</td>
<td>0.760</td>
<td>( &lt; 10^{-3} )</td>
<td>0.56</td>
</tr>
<tr>
<td>Heusner</td>
<td>( 34 )</td>
<td>0.877</td>
<td>( &lt; 10^{-12} )</td>
<td>( &lt; 10^{-7} )</td>
</tr>
</tbody>
</table>
Fluctuations—Kolmogorov-Smirnov test

Analysis of residuals

We use the spiffing Spearman Rank-Order Correlation Coefficient.

Basic idea:

- Given \( \{(x_i, y_i)\} \), rank the \( \{x_i\} \) and \( \{y_i\} \) separately from smallest to largest. Call these ranks \( R_i \) and \( S_i \).
- Now calculate correlation coefficient for ranks, \( r_s \):
  \[
  r_s = \frac{\sum_{i=1}^{n}(R_i - \bar{R})(S_i - \bar{S})}{\sqrt{\sum_{i=1}^{n}(R_i - \bar{R})^2} \sqrt{\sum_{i=1}^{n}(S_i - \bar{S})^2}}
  \]
- Perfect correlation: \( x_i \)'s and \( y_i \)'s both increase monotonically.

Analysis of residuals

We assume all rank orderings are equally likely:

- \( r_s \) is distributed according to a Student's distribution with \( N - 2 \) degrees of freedom.
- Excellent feature: Non-parametric—real distribution of \( x \)'s and \( y \)'s doesn't matter.
- Bonus: works for non-linear monotonic relationships as well.
- See “Numerical Recipes in C/Fortran” which contains many good things. [20]
Analysis of residuals—mammals

(a) $M < 3.2$ kg, (b) $M < 10$ kg, (c) $M < 32$ kg, (d) all mammals.

Analysis of residuals—birds

(a) $M < 0.1$ kg, (b) $M < 1$ kg, (c) $M < 10$ kg, (d) all birds.

Basic basin quantities: $a$, $l$, $L_{\parallel}$, $L_{\perp}$:

- $a =$ drainage basin area
- $l =$ length of longest (main) stream
- $L = L_{\parallel} =$ longitudinal length of basin

River networks

1957: J. T. Hack\cite{11}

“Studies of Longitudinal Stream Profiles in Virginia and Maryland”

\[ \ell \sim a^h \]

\[ h \sim 0.6 \]

- Anomalous scaling: we would expect $h = 1/2$...
- Subsequent studies: $0.5 \lesssim h \lesssim 0.6$
- Another quest to find universality/god...
- A catch: studies done on small scales.
Large-scale networks

(1992) Montgomery and Dietrich \[^{19}\]:

- Composite data set: includes everything from unchanneled valleys up to world’s largest rivers.
- Estimated fit:
  \[ L \approx 1.78a^{0.49} \]
- Mixture of basin and main stream lengths.

Earlier theories

Building on the surface area idea...

- Blum (1977) \[^{5}\] speculates on four-dimensional biology:
  \[ P \propto M^{(d-1)/d} \]
- \( d = 3 \) gives \( \alpha = 2/3 \)
- \( d = 4 \) gives \( \alpha = 3/4 \)
- So we need another dimension...
- Obviously, a bit silly.

World’s largest rivers only:

- Data from Leopold (1994) \[^{16, 8}\]
- Estimate of Hack exponent: \( h = 0.50 \pm 0.06 \)

Earlier theories

Building on the surface area idea:

- McMahon (70’s, 80’s): Elastic Similarity \[^{17, 18}\]
- Idea is that organismal shapes scale allometrically with 1/4 powers (like nails and trees...)
- Appears to be true for ungulate legs.
- Metabolism and shape never properly connected.
Nutrient delivering networks:

- 1960's: Rashevsky considers blood networks and finds a $2/3$ scaling.

**West et al.'s assumptions:**

- hierarchical network
- capillaries (delivery units) invariant
- network impedance is minimized via evolution

**Claims:**

- $P \propto M^{3/4}$
- networks are fractal
- quarter powers everywhere

Impedance measures:

Poiseuille flow (outer branches):

$$Z = \frac{8\mu}{\pi} \sum_{k=0}^{N} \frac{\ell_k}{r_k^4 N_k}$$

Pulsatille flow (main branches):

$$Z \propto \sum_{k=0}^{N} \frac{h_k^{1/2}}{r_k^{5/2} N_k}$$

Not so fast . . .

Actually, model shows:

- $P \propto M^{3/4}$ does not follow for pulsatile flow
- networks are not necessarily fractal.

Do find:

- Murray’s cube law (1927) for outer branches:
  $$r_0^3 = r_1^3 + r_2^3$$
  - Impedance is distributed evenly.
  - Can still assume networks are fractal.
Connecting network structure to $\alpha$

1. Ratios of network parameters:
   \[ R_n = \frac{n_{k+1}}{n_k}, R_\ell = \frac{\ell_{k+1}}{\ell_k}, R_r = \frac{r_{k+1}}{r_k} \]

2. Number of capillaries $\propto P \propto M^\alpha$.
   \[ \Rightarrow \alpha = -\frac{\ln R_n}{\ln R_\ell^2 R_\ell} \]
   (also problematic due to prefactor issues)

Soldiering on, assert:

- area-preservingness: $R_r = R_n^{-1/2}$
- space-fillingness: $R_\ell = R_n^{-1/3}$

\[ \Rightarrow \alpha = 3/4 \]

Data from real networks

<table>
<thead>
<tr>
<th>Network</th>
<th>$R_n$</th>
<th>$R_\ell^{-1}$</th>
<th>$R_r^{-1}$</th>
<th>$-\ln R_n$</th>
<th>$-\ln R_\ell$</th>
<th>$-\ln R_r$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>West et al.</td>
<td></td>
<td></td>
<td></td>
<td>1/2</td>
<td>1/3</td>
<td></td>
<td>3/4</td>
</tr>
<tr>
<td>rat (PAT)</td>
<td>2.76</td>
<td>1.58</td>
<td>1.60</td>
<td>0.45</td>
<td>0.46</td>
<td>0.73</td>
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<tr>
<td>cat (PAT)</td>
<td>3.67</td>
<td>1.71</td>
<td>1.78</td>
<td>0.41</td>
<td>0.44</td>
<td>0.79</td>
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<tr>
<td>(Turcotte et al.[24])</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dog (PAT)</td>
<td>3.69</td>
<td>1.67</td>
<td>1.52</td>
<td>0.39</td>
<td>0.32</td>
<td>0.90</td>
<td></td>
</tr>
<tr>
<td>pig (LCX)</td>
<td>3.57</td>
<td>1.89</td>
<td>2.20</td>
<td>0.50</td>
<td>0.62</td>
<td>0.62</td>
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<tr>
<td>pig (RCA)</td>
<td>3.50</td>
<td>1.81</td>
<td>2.12</td>
<td>0.47</td>
<td>0.60</td>
<td>0.65</td>
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<tr>
<td>pig (LAD)</td>
<td>3.51</td>
<td>1.84</td>
<td>2.02</td>
<td>0.49</td>
<td>0.56</td>
<td>0.65</td>
<td></td>
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<tr>
<td>human (PAT)</td>
<td>3.03</td>
<td>1.60</td>
<td>1.49</td>
<td>0.42</td>
<td>0.36</td>
<td>0.83</td>
<td></td>
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<tr>
<td>human (PAT)</td>
<td>3.36</td>
<td>1.56</td>
<td>1.49</td>
<td>0.37</td>
<td>0.33</td>
<td>0.94</td>
<td></td>
</tr>
</tbody>
</table>

Simple supply networks

- Banavar et al., Nature, (1999)\[1\]
- Flow rate argument
- Ignore impedance
- Very general attempt to find most efficient transportation networks

Simple supply networks

- Banavar et al. find ‘most efficient’ networks with
  \[ P \propto M^{d/(d+1)} \]
- ... but also find
  \[ V_{\text{network}} \propto M^{(d+1)/d} \]
- $d = 3$:
  \[ V_{\text{blood}} \propto M^{4/3} \]
- Consider a 3 g shrew with $V_{\text{blood}} = 0.1 V_{\text{body}}$
- \[ \Rightarrow 3000 \text{ kg elephant with } V_{\text{blood}} = 10 V_{\text{body}} \]
- Such a pachyderm would be rather miserable.
Geometric argument

- Consider one source supplying many sinks in a \(d\)-dim. volume in a \(D\)-dim. ambient space.
- Assume sinks are invariant.
- Assume \(p = \rho \langle V \rangle\).
- Assume some cap on flow speed of material.
- See network as a bundle of virtual vessels:

![Diagram of network with source and sinks]

- Q: how does the number of sustainable sinks \(N_{\text{sinks}}\) scale with volume \(V\) for the most efficient network design?
- Or: what is the highest \(\alpha\) for \(N_{\text{sinks}} \propto V^\alpha\)?

Geometric argument

- Best and worst configurations (Banavar et al.)

![Diagram of networks]

- Rather obviously:
  \[\min V_{\text{net}} \propto \sum \text{distances from source to sinks}.\]

Geometric argument

- Allometrically growing regions:

![Diagram of network growth]

- Have \(d\) length scales which scale as

\[L_i \propto V^{\gamma_i}\]

where \(\gamma_1 + \gamma_2 + \ldots + \gamma_d = 1\).
- For isometric growth, \(\gamma_i = 1/d\).
- For allometric growth, we must have at least two of the \(\{\gamma_i\}\) being different.

Minimal network volume:

Real supply networks are close to optimal:

![Diagram of networks]

\[\text{Figure 1. (a) Commuter rail network in the Boston area. The arrow marks the assumed root of the network. (b) Star graph. (c) Minimum spanning tree. (d) The model of equation (3) applied to the same set of stations.}\]

(2006) Gastner and Newman\cite{10}: “Shape and efficiency in spatial distribution networks”
Minimal network volume:

Approximate network volume by integral over region:

\[
\min V_{\text{net}} \propto \int_{\Omega_d, D(V)} \rho \left| \vec{x} \right| d\vec{x}
\]

\[
\rightarrow \rho V^{1+\gamma_{\text{max}}} \int_{\Omega_d, D(c)} \left( c_1^2 u_1^2 + \ldots + c_k^2 u_k^2 \right)^{1/2} d\vec{u}
\]

\[
\propto \rho V^{1+\gamma_{\text{max}}}
\]

Blood networks

- Material costly \( \Rightarrow \) expect lower optimal bound of \( V_{\text{net}} \propto \rho V^{(d+1)/d} \) to be followed closely.

- For cardiovascular networks, \( d = D = 3 \).

- Blood volume scales linearly with body volume \([23]\), \( V_{\text{net}} \propto V \).

- Sink density must \( \downarrow \) decrease as volume increases:

\[
\rho \propto V^{-1/d}
\]

- Density of suppliable sinks decreases with organism size.

Geometric argument

- General result:

\[
\min V_{\text{net}} \propto \rho V^{1+\gamma_{\text{max}}}
\]

- If scaling is isometric, we have \( \gamma_{\text{max}} = 1/d \):

\[
\min V_{\text{net/iso}} \propto \rho V^{1+1/d} = \rho V^{(d+1)/d}
\]

- If scaling is allometric, we have \( \gamma_{\text{max}} = \gamma_{\text{allo}} > 1/d \) and

\[
\min V_{\text{net/allo}} \propto \rho V^{1+\gamma_{\text{allo}}}
\]

- Isometrically growing volumes require less network volume than allometrically growing volumes:

\[
\frac{\min V_{\text{net/iso}}}{\min V_{\text{net/allo}}} \rightarrow 0 \text{ as } V \rightarrow \infty
\]

Blood networks

- Then \( P \), the rate of overall energy use in \( \Omega \), can at most scale with volume as

\[
P \propto \rho V \propto \rho M \propto M^{(d-1)/d}
\]

- For \( d = 3 \) dimensional organisms, we have

\[
P \propto M^{2/3}
\]
Recap:

- The exponent $\alpha = 2/3$ works for all birds and mammals up to $10–30$ kg
- For mammals $> 10–30$ kg, maybe we have a new scaling regime
- Economos: limb length break in scaling around 20 kg
- White and Seymour, 2005: unhappy with large herbivore measurements. Find $\alpha \simeq 0.686 \pm 0.014$

River networks

- View river networks as collection networks.
- Many sources and one sink.
- Assume $\rho$ is constant over time:
  \[ V_{\text{net}} \propto \rho V^{(d+1)/d} = \text{constant} \times V^{3/2} \]
- Network volume grows faster than basin ‘volume’ (really area).
- It’s all okay: Landscapes are $d=2$ surfaces living in $D=3$ dimension.
- Streams can grow not just in width but in depth...

Hack’s law

- Volume of water in river network can be calculated by adding up basin areas
- Flows sum in such a way that:
  \[ V_{\text{net}} = \sum_{\text{all pixels}} a_{\text{pixel } i} \]
- Hack’s law again:
  \[ \ell \sim a^h \]
- Can argue:
  \[ V_{\text{net}} \propto V^{1+h}_{\text{basin}} = a_{\text{basin}}^{1+h} \]
  where $h$ is Hack’s exponent.
- ∴ minimal volume calculations gives
  \[ h = 1/2 \]

Prefactor:

Stefan-Boltzmann law:

- \[ \frac{dE}{dt} = \sigma S T^4 \]
  where $S$ is surface and $T$ is temperature.
- Very rough estimate of prefactor based on scaling of normal mammalian body temperature and surface area $S$:
  \[ B \approx 10^5 M^{2/3} \text{erg/sec} \]
- Measured for $M \leq 10$ kg:
  \[ B = 2.57 \times 10^5 M^{2/3} \text{erg/sec} \]
Real data:

- Banavar et al.'s approach\(^1\) is okay because \(\rho\) really is constant.
- The irony: shows optimal basins are isometric.
- Optimal Hack’s law: \(\ell \sim a^h\) with \(h = 1/2\).
- (Zzzzz)

From Banavar et al. (1999)\(^1\)

Even better—prefactors match up:

<table>
<thead>
<tr>
<th>River</th>
<th>Water Volume [m(^3)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amazon</td>
<td>20</td>
</tr>
<tr>
<td>Mississippi</td>
<td>18</td>
</tr>
<tr>
<td>Congo</td>
<td>15</td>
</tr>
<tr>
<td>Nile</td>
<td>12</td>
</tr>
</tbody>
</table>

\(\log_{10} \text{area } a \text{ [m}^2\text{]}\)

\(\log_{10} \text{water volume } V \text{ [m}^3\text{]}\)

Conclusion

- Supply network story consistent with dimensional analysis.
- Isometrically growing regions can be more efficiently supplied than allometrically growing ones.
- Ambient and region dimensions matter \((D = d \text{ versus } D > d)\).
- Deviations from optimal scaling suggest inefficiency (e.g., gravity for organisms, geological boundaries).
- Actual details of branching networks not that important.
- Exact nature of self-similarity varies.

References I

Scaling

Scaling-at-large

Allometry

Definitions

History; Metabolism

Measuring exponents

History; River networks

Earlier theories

Geometric argument

Blood networks

River networks

Conclusion

References

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