Lecture 3 (Chapter 2)
Linear Algebra, Course 124A, Fall, 2009

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Solving $A\vec{x} = \vec{b}$:

- We (people + computers) solve systems of linear equations by a systematic method of **Elimination** followed by **Back substitution**
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

\[-x_1 + 3x_2 = 1\]
\[2x_1 + x_2 = 5\]
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Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.
2. Swap rows if needed to create an ‘upper triangular form’
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e.g.

$$2x_1 - x_2 = -1 \implies 2x_1 - x_2 = -1 \quad \Rightarrow \quad x_2 = 3$$
Gaussian elimination:

Solve:

\[2x_1 - 3x_2 = 3\]
\[4x_1 - 5x_2 + x_3 = 7\]
\[2x_1 - x_2 - 3x_3 = 5\]
Gaussian elimination:

Summary:
Using row operations, we turned this problem:

\[
A \vec{x} = \vec{b} : \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}
\]

into this problem:

\[
U \vec{x} = \vec{d} : \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}
\]

and the latter is easy to solve using back substitution.
Gaussian elimination:

Defn:
The entries along \( U \)'s main diagonal are the \textit{pivots} of \( A \).
(The pivots are hidden—elimination finds them.)

Defn:
A matrix with only zeros below the main diagonal is called \textit{upper triangular}. A matrix with only zeros above the main diagonal is called \textit{lower triangular}. We get from \( A \) to \( U \) and the latter is always upper triangular.

Defn:
\textbf{Singular} means a system has no unique solution.

- It may have no solutions or infinitely many solutions.
- \textit{Singular} = archaic way of saying ‘messed up.’

Truth:
If at least one pivot is zero, the matrix will be \textit{singular}.
(but the reverse is not necessarily true).
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Gaussian elimination:

The one true method:

- We simplify $A$ using elimination in the same way every time.
- Eliminate entries one column at a time, moving left to right, and down each column.

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\begin{align*}
X + X + X + X &= X \\
1 \downarrow + X + X + X &= X \\
2 \downarrow + 4 \downarrow + X + X &= X \\
3 \nearrow + 5 \rightarrow + 6 + X &= X 
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Gaussian elimination:

- To eliminate entry in row \(i\) of \(j\)th column, subtract a multiple \(\ell_{ij}\) of the \(j\)th row from \(i\).
- For example:

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\begin{align*}
2x_1 + 3x_2 - 2x_3 + x_4 &= 1 \\
x_1 - 7x_2 + 3x_3 + x_4 &= 1 \\
-x_1 - 3x_2 - x_3 + 5x_4 &= -2 \\
2x_1 + x_2 - 2x_3 + 2x_4 &= 0
\end{align*}
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\(\ell_{21} = 1/2, \ell_{31} = -1/2, \ell_{41} = ?\).
- Note: we cannot find \(\ell_{32}\) etc., until we are finished with row 1. Pivots are hidden!
- Note: the denominator of each \(\ell_{ij}\) multiplier is the pivot in the \(j\)th column.
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