Lecture 3 (Chapter 2)
Linear Algebra, Course 124A, Fall, 2009

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Solving $A\vec{x} = \vec{b}$:

- We (people + computers) solve systems of linear equations by a systematic method of Elimination followed by Back substitution.
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

  
  
  
  \[-x_1 + 3x_2 = 1
  
  2x_1 + x_2 = 5

  


Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.
2. Swap rows if needed to create an ‘upper triangular form’

E.g.

\[
\begin{align*}
2x_1 & - x_2 = -1 \\
\Rightarrow & \\
\Rightarrow x_2 & = 3 \\
\Rightarrow 2x_1 & - x_2 = -1 \\
2x_1 & - x_2 = 3
\end{align*}
\]
Gaussian elimination:

Solve:

\[ 2x_1 - 3x_2 = 3 \]
\[ 4x_1 - 5x_2 + x_3 = 7 \]
\[ 2x_1 - x_2 - 3x_3 = 5 \]
Gaussian elimination:

Summary:
Using row operations, we turned this problem:

\[
A\vec{x} = \vec{b} : \begin{bmatrix}
2 & -3 & 0 \\
4 & -5 & 1 \\
2 & -1 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
3 \\
7 \\
5
\end{bmatrix}
\]

into this problem:

\[
U\vec{x} = \vec{d} : \begin{bmatrix}
2 & -3 & 0 \\
0 & 1 & 1 \\
0 & 0 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
3 \\
1 \\
0
\end{bmatrix}
\]

and the latter is easy to solve using back substitution.
Gaussian elimination:

Defn:
The entries along $U$'s main diagonal are the pivots of $A$. (The pivots are hidden—elimination finds them.)

Defn:
A matrix with only zeros below the main diagonal is called upper triangular. A matrix with only zeros above the main diagonal is called lower triangular. We get from $A$ to $U$ and the latter is always upper triangular.

Defn:
Singular means a system has no unique solution.
- It may have no solutions or infinitely many solutions.
- Singular = archaic way of saying ‘messed up.’

Truth:
If at least one pivot is zero, the matrix will be singular. (but the reverse is not necessarily true).
Gaussian elimination:

The one true method:

- We simplify $A$ using elimination in the same way every time.
- Eliminate entries one column at a time, moving left to right, and down each column.

\[
\begin{align*}
X + X + X + X &= X \\
1 \downarrow &+ X + X + X = X \\
2 \downarrow &+ 4 \downarrow + X + X = X \\
3 \nearrow &+ 5 \rightarrow + 6 + X = X
\end{align*}
\]
Gaussian elimination:

- To eliminate entry in row $i$ of $j$th column, subtract a multiple $\ell_{ij}$ of the $j$th row from $i$.

- For example:

\[
\begin{align*}
2x_1 + 3x_2 + -2x_3 + x_4 &= 1 \\
x_1 - 7x_2 + 3x_3 + x_4 &= 1 \\
-x_1 - 3x_2 - x_3 + 5x_4 &= -2 \\
2x_1 + x_2 - 2x_3 + 2x_4 &= 0
\end{align*}
\]

$\ell_{21} = 1/2, \; \ell_{31} = -1/2, \; \ell_{41} = ?$.

- **Note:** we cannot find $\ell_{32}$ etc., until we are finished with row 1. Pivots are hidden!

- **Note:** the denominator of each $\ell_{ij}$ multiplier is the pivot in the $j$th column.