Random Networks
Complex Networks, Course 303A, Spring, 2009

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Outline

Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Largest component

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- Average Component Size

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Random networks

Pure, abstract random networks:

- Consider set of all networks with \( N \) labelled nodes and \( m \) edges.
- Standard random network = \textit{randomly chosen} network from this set.
- To be clear: each network is \textit{equally} probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or \textit{ER} graphs.
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Some features:

- Number of possible edges:

\[ 0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2} \]

- Given \( m \) edges, there are \( \binom{N}{m} \) different possible networks.
- Crazy factorial explosion for \( 1 \ll m \ll \binom{N}{2} \).
- Limit of \( m = 0 \): empty graph.
- Limit of \( m = \binom{N}{2} \): complete or fully-connected graph.
- Real world: links are usually costly so real networks are almost always sparse.
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How to build standard random networks:

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability $p$.
   - Useful for theoretical work.

2. Take $N$ nodes and add exactly $m$ links by selecting edges without replacement.
   - Algorithm: Randomly choose a pair of nodes $i$ and $j$, $i \neq j$, and connect if unconnected; repeat until all $m$ edges are allocated.
   - Best for adding small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large $N$. 

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- Given $N$ and $m$.
- Two probabilistic methods (we’ll see a third later on)

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A few more things:

- For method 1, # links is probabilistic:

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\langle m \rangle = p\binom{N}{2} = p\frac{1}{2}N(N - 1)
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- So the expected or average degree is

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Random networks: examples

Next slides:
Example realizations of random networks

- $N = 500$
- Vary $m$, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.
Random networks: examples

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Example realizations of random networks

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entire network:

largest component:

\[ N = 500, \text{ number of edges } m = 100 \]
\[ \langle k \rangle = 0.4 \]
Random networks: examples

entire network:

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\[ N = 500, \text{ number of edges } m = 200 \]
\[ \text{average degree } \langle k \rangle = 0.8 \]
Random networks: examples

entire network:  
largest component:

\[ N = 500, \text{ number of edges } m = 230 \]
\[ \langle k \rangle = 0.92 \]
Random networks: examples

entire network:

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\[ N = 500, \text{ number of edges } m = 240 \]
\[ \langle k \rangle = 0.96 \]
Random networks: examples

entire network:

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\[ N = 500, \text{ number of edges } m = 250 \]
\[ \langle k \rangle = 1 \]
Random networks: examples

entire network:  
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$N = 500$, number of edges $m = 260$
average degree $\langle k \rangle = 1.04$
Random networks: examples

entire network:

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\( N = 500, \) number of edges \( m = 280 \)
average degree \( \langle k \rangle = 1.12 \)
Random networks: examples

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\[ N = 500, \text{ number of edges } m = 300, \text{ average degree } \langle k \rangle = 1.2 \]
Random networks: examples

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\[ N = 500, \text{ number of edges } m = 500 \]
\[ \text{average degree } \langle k \rangle = 2 \]
Random networks: examples

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\[ \langle k \rangle = 4 \]
Random networks: examples for $N=500$

- $m = 100 \langle k \rangle = 0.4$
- $m = 120 \langle k \rangle = 0.8$
- $m = 230 \langle k \rangle = 0.92$
- $m = 240 \langle k \rangle = 0.96$
- $m = 250 \langle k \rangle = 1$
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- $m = 280 \langle k \rangle = 1.12$
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Clustering:

- For method 1, what is the clustering coefficient for a finite network?

- Consider triangle/triple clustering coefficient (Newman [1]):

\[ C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}} \]

- Recall: \( C_2 \) = probability that two nodes are connected given they have a friend in common.

- For standard random networks, we have simply that

\[ C_2 = p. \]
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- So for large random networks \((N \to \infty)\), clustering drops to zero.

- Key structural feature of random networks is that they locally look like branching networks (no loops).
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Degree distribution:

- Recall $p_k$ = probability that a randomly selected node has degree $k$.
- Consider method 1 for constructing random networks: each possible link is realized with probability $p$.
- Now consider one node: there are ‘$N - 1$ choose $k$’ ways the node can be connected to $k$ of the other $N - 1$ nodes.
- Each connection occurs with probability $p$, each non-connection with probability $(1 - p)$.
- Therefore have a binomial distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1 - p)^{N-1-k}.$$
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Limiting form of $P(k; p, N)$:

- Our degree distribution:
  
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- What happens as $N \rightarrow \infty$?
- We must end up with the normal distribution right?
- If $p$ is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \approx pN \rightarrow \infty$.
- But we want to keep $\langle k \rangle$ fixed...
- So examine limit of $P(k; p, N)$ when $p \rightarrow 0$ and $N \rightarrow \infty$ with $\langle k \rangle = p(N - 1) = \text{constant}$. 

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$$P(k; p, N) \sim \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N - 1}\right)^{N-1-k}$$

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$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$ (Use l'Hôpital's rule to prove.)

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General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution $P_k$.
- Also known as the configuration model \([1]\).
- Can generalize construction method from ER random networks.
- Assign each node a weight $w$ from some distribution $P_w$ and form links with probability
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P(\text{link between } i \text{ and } j) \propto w_i w_j.
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- But we’ll be more interested in
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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

- $N = 1000$.
- $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- Set $P_0 = 0$ (no isolated nodes).
- Vary exponent $\gamma$ between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.
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<th>$\gamma$</th>
<th>$\langle k \rangle$</th>
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- Normalization: we must have

\[ \sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1 \]

- Checking:

\[ \sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \]

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- The variance of degree distributions for random networks turns out to be very important.
- Use calculation similar to one for finding $\langle k \rangle$ to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

- So standard deviation $\sigma$ is equal to $\sqrt{\langle k \rangle}$.
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- The degree distribution $P_k$ is fundamental for our description of many complex networks.
- Again: $P_k$ is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define $Q_k$ to be the probability the node at a random end of a randomly chosen edge has degree $k$.
- Now choosing nodes based on their degree (i.e., size):
  \[ Q_k \propto kP_k \]
- Normalized form:
  \[ Q_k = \frac{kP_k}{\sum_{k'=0}^{\infty} k'P_{k'}} = \frac{kP_k}{\langle k \rangle}. \]
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- For random networks, $Q_k$ is also the probability that a friend (neighbor) of a random node has $k$ friends.
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- Equivalent to friend having degree $k + 1$.
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Reason #1:

- Average # friends of friends per node is

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- Key: Average depends on the 1st and 2nd moments of $P_k$ and not just the 1st moment.

- Three peculiarities:
  1. We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it’s actually $\langle k(k - 1) \rangle.$
  2. If $P_k$ has a large second moment, then $\langle k_2 \rangle$ will be big. (e.g., in the case of a power-law distribution)
  3. Your friends are different to you...
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- A node’s average # of friends: $\langle k \rangle$
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$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle \cdot \frac{\langle k^2 \rangle}{\langle k \rangle^2}}{\langle k \rangle^2} = \langle k \rangle \cdot \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2}\right) \geq \langle k \rangle$$

- So only if everyone has the same degree (variance $= \sigma^2 = 0$) can a node be the same as its friends.
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(Big) Reason #2:

- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we’d like to know what’s the size of the largest component within a network.
- As $N \to \infty$, does our network have a giant component?
- Defn: Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- Defn: Giant component = component that comprises a non-zero fraction of a network as $N \to \infty$.
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Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.

- Equivalently, expect exponential growth in node number as we move out from a random node.

- All of this is the same as requiring \( \langle k \rangle_R > 1 \).

- Giant component condition (or percolation condition):

\[
\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1
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- Again, see that the second moment is an essential part of the story.

- Equivalent statement: \( \langle k^2 \rangle > 2\langle k \rangle \)
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> Recall \( \langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle \).

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> When \( \langle k \rangle < 1 \), all components are finite.

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Random networks with skewed $P_k$:

- e.g., if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$ then

$$\langle k^2 \rangle = c \sum_{k=0}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=0}^{\infty} x^{2-\gamma} dx$$

$$\propto x^{3-\gamma} \bigg|_{x=0}^{\infty} = \infty \quad (> \langle k \rangle).$$

- So giant component always exists for these kinds of networks.

- Cutoff scaling is $k^{-3}$: if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$. 
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And how big is the largest component?

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- Consider an infinite ER random network with average degree $\langle k \rangle$.
- Let’s find $S_1$ with a back-of-the-envelope argument.
- Define $\delta$ as the probability that a randomly chosen node does not belong to the largest component.
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Carrying on:

\[ \delta = \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \]

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\[ = e^{-\langle k \rangle} e^{\langle k \rangle} \delta = e^{-\langle k \rangle} (1 - \delta). \]

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- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of $S_1$:
  $$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$  
- As $\langle k \rangle \to 0$, $S_1 \to 0$.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$.
- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S > 0$ when $\langle k \rangle > 1$.
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- Our dirty trick only works for ER random networks.
  - The problem: We assumed that neighbors have the same probability $\delta$ of belonging to the largest component.
  - But we know our friends are different from us...
  - Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
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Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Largest component

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- Average Component Size

References
Generating functions

**Idea:** Given a sequence $a_0, a_1, a_2, \ldots$, associate each element with a distinct function or other mathematical object.

**Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.**

**Definition:**

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$ 

- Roughly: transforms a vector in $\mathbb{R}^\infty$ into a function defined on $\mathbb{R}^1$.

- Related to Fourier, Laplace, Mellin, ...
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Simple example

Rolling dice:

- $p_k^{(□)} = P_r(\text{throwing a } k) = \frac{1}{6}$ where $k = 1, 2, \ldots, 6$.

- $F^{(□)}(x) = \sum_{k=1}^{6} p_k x^k = \frac{1}{6} (x + x^2 + x^3 + x^4 + x^5 + x^6)$.

- We’ll come back to this simple example as we derive various delicious properties of generating functions.
Simple example

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Example

- Take a degree distribution with exponential decay:

\[ P_k = ce^{-\lambda k} \]

where \( c = 1 - e^{-\lambda} \).

- The generating function for this distribution is

\[ F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}. \]

- Notice that \( F(1) = c/(1 - e^{-\lambda}) = 1 \).

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- Largest component

Generating Functions
- Definitions
- Basic Properties
  - Giant Component Condition
  - Component sizes
  - Useful results
  - Size of the Giant Component
  - Average Component Size

References
Properties of generating functions

- **Average degree:**

\[
\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \left. \sum_{k=0}^{\infty} k P_k x^{k-1} \right|_{x=1} = \frac{d}{dx} F(x) \bigg|_{x=1} = F'(1)
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F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.
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Useful pieces for probability distributions:

- **Normalization:**
  \[ F(1) = 1 \]

- **First moment:**
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- **Higher moments:**
  \[ \langle k^n \rangle = \left( x \frac{d}{dx} \right)^n F(x) \bigg|_{x=1} \]

- **kth element of sequence (general):**
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We first need the g.f. for \( R_k \).

We’ll now use this notation:

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We have

\[ F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k + 1) P_{k+1}}{\langle k \rangle} x^k. \]

Shift index to \( j = k + 1 \) and pull out \( \frac{1}{\langle k \rangle} \):

\[ F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{d}{dx} x^j \]

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- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
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- Useful results
- Size of the Giant Component
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To figure out the size of the largest component ($S_1$), we need more resolution on component sizes.

Definitions:

- $\pi_n$ = probability that a random node belongs to a finite component of size $n < \infty$.
- $\rho_n$ = probability a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

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The largest component:

- Subtle key: \( F_\pi(1) \) is the probability that a node belongs to a finite component.
- Therefore: \( S_1 = 1 - F_\pi(1) \).

Our mission, which we accept:

- Find the four generating functions \( F_P, F_R, F_\pi, \) and \( F_\rho \).
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Our mission, which we accept:

- Find the four generating functions \( F_P, F_R, F_\pi, \) and \( F_\rho \).
Size distributions

G.f.'s for component size distributions:

\[ F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \quad \text{and} \quad F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n \]

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Sneaky Result 1:

- Consider two random variables $U$ and $V$ whose values may be 0, 1, 2, ...
- Write probability distributions as $U_k$ and $U_k$ and g.f.’s as $F_U$ and $F_V$.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{V} U^{(i)}$$

with each $U^{(i)} \overset{d}{=} U$

then

$$F_W(x) = F_V(F_U(x))$$
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With some concentration, observe:

\[ F_W(x) = \sum_{j=0}^{\infty} V_j \sum_{k=0}^{\infty} \sum_{\{i_1, i_2, \ldots, i_j\} \mid i_1 + i_2 + \ldots + i_j = k} U_{i_1} x^{i_1} U_{i_2} x^{i_2} \cdots U_{i_j} x^{i_j} \]

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\[ x^k \text{ piece of } \left( \sum_{i'=0}^{\infty} U_{i'} x^{i'} \right)^j \]

\[ \left( \sum_{i'=0}^{\infty} U_{i'} x^{i'} \right)^j = (F_U(x))^j \]

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- Start with a random variable \( U \) with distribution \( U_k \) (\( k = 0, 1, 2, \ldots \))
- **SR2**: If a second random variable is defined as \( V = U + 1 \) then \( F_V(x) = xF_U(x) \)

Reason: \( V_k = U_{k-1} \) for \( k \geq 1 \) and \( V_0 = 0 \).

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Generalization of SR2:

1. If $V = U + i$ then
   \[ F_V(x) = x^i F_U(x). \]

2. If $V = U - i$ then
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Connecting generating functions

- **Goal:** figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

- $\pi_n$ = probability that a random node belongs to a finite component of size $n$

\[
\pi_n = \sum_{k=0}^{\infty} P_k \times \Pr \left( \text{sum of sizes of subcomponents at end of k random links} = n - 1 \right)
\]

- Therefore:

\[
F_\pi(x) = x \frac{F_\rho(F_\rho(x))}{SR1}
\]

- Extra factor of $x$ accounts for random node itself.
Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

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Connecting generating functions

- **Goal:** figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

- $\pi_n$ = probability that a random node belongs to a finite component of size $n$

\[
\pi_0 = \sum_{k=0}^{\infty} P_k \times \Pr \left( \text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1 \right)
\]

Therefore:

\[
F_\pi(x) = x \left( 1 - F_\rho \left( F_\pi(x) \right) \right)
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- Extra factor of $x$ accounts for random node itself.
Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

$p_n = \text{probability that a random node belongs to a finite component of size } n$

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Therefore: $F_\pi(x) = \frac{x}{SR2} F_\rho(F_\rho(x))$

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$$= \sum_{k=0}^{\infty} P_k \times \Pr\left(\text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1\right)$$

Therefore:

$$F_\pi(x) = x \underbrace{F_P(F_\rho(x))}_{SR2} \underbrace{SR1}_{SR}$$

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Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

$\pi_n =$ probability that a random node belongs to a finite component of size $n$

$$= \sum_{k=0}^{\infty} P_k \times \text{Pr}\left(\text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1\right)$$

Therefore:

$$F_\pi(x) = x \left[ F_P(F_\rho(x)) \right]_{SR2}^{SR1}$$

Extra factor of $x$ accounts for random node itself.
Connecting generating functions

- $\rho_n = \text{probability that a random link leads to a finite subcomponent of size } n.$

- Invoke one step of recursion: $\rho_n = \text{probability that a random node arrived along a random edge is part of a finite subcomponent of size } n.$

\[
\rho_n = \sum_{k=0}^{\infty} R_k \times \text{Pr} \left( \text{sum of sizes of subcomponents at end of } k \text{ random links} = n - 1 \right)
\]

Therefore:

\[
F_\rho(x) = \frac{x}{SR2} \left( \frac{F_R(F_\rho(x))}{SR1} \right)
\]

- Again, extra factor of $x$ accounts for random node itself.
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- $\rho_n$ = probability that a random link leads to a finite subcomponent of size $n$.
- Invoke one step of recursion: $\rho_n$ = probability that a random node arrived along a random edge is part of a finite subcomponent of size $n$.

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left( \text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1 \right)$$

Therefore:

$$F_\rho(x) = x \overbrace{F_R(F_\rho(x))}^{SR1} \overbrace{\sum_{k=0}^{\infty} R_k}^{SR2}$$

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\[
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\]

Therefore:

\[
F_\rho(x) = x \frac{F_R(F_\rho(x))}{SR2} \frac{1}{SR1}
\]

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- Invoke one step of recursion: \( \rho_n \) = probability that a random node arrived along a random edge is part of a finite subcomponent of size \( n \).

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Therefore:

\[
F_{\rho}(x) = x \frac{F_R(\rho(x))}{SR1} + \frac{x}{SR2}
\]

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Connecting generating functions

- We now have two functional equations connecting our generating functions:

\[ F_\pi(x) = xF_P(F_\rho(x)) \quad \text{and} \quad F_\rho(x) = xF_R(F_\rho(x)) \]

- Taking stock: We know \( F_P(x) \) and \( F_R(x) = F'_P(x)/F'_P(1) \).
- We first untangle the second equation to find \( F_\rho \).
- We can do this because it only involves \( F_\rho \) and \( F_R \).
- The first equation then immediately gives us \( F_\pi \) in terms of \( F_\rho \) and \( F_R \).
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Component sizes

- Remembering vaguely what we are doing:
  - Finding $F_\pi$ to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.
  - Set $x = 1$ in our two equations:
    \[ F_\pi(1) = F_P(F_\rho(1)) \quad \text{and} \quad F_\rho(1) = F_R(F_\rho(1)) \]

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Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

传导: $F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x' = 1}$

\[= e^{-\langle k \rangle (1-x)} = F_P(x) \quad \text{...aha!} \]

- RHS’s of our two equations are the same.

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  \[ F_\pi(x) = xF_R(F_\pi(x)) \] and \[ F_R(x) = e^{-\langle k \rangle (1-x)}. \]

- \[ \therefore F_\pi(x) = xe^{-\langle k \rangle (1-F_\pi(x))} \]

- We’re first after \( S_1 = 1 - F_\pi(1) \) so set \( x = 1 \) and replace \( F_\pi(1) \) by \( 1 - S_1 \):

  \[ 1 - S_1 = e^{-\langle k \rangle S_1} \]

Or: \( \langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1} \)

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Outline

Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Largest component

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useless results
- Size of the Giant Component

Average Component Size

References
Next: find **average size** of finite components $\langle n \rangle$.

Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.

Try to avoid finding $F_\pi(x)$...

Starting from $F_\pi(x) = xF_P(F_\rho(x))$, we differentiate:

$$F'_\pi(x) = F_P(F_\rho(x)) + xF'_\rho(x)F'_P(F_\rho(x))$$

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Now set $x = 1$ in both equations.

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$$F'_\pi(x) = F_P(F_\rho(x)) + xF'_\rho(x)F'_P(F_\rho(x))$$

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- Now set $x = 1$ in both equations.
- We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).
- Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.
Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
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Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P(F_\pi(x)) + xF'_\pi(x)F'_P(F_\pi(x))$$

Rearrange:

$$F'_\pi(x) = \frac{F_P(F_\pi(x))}{1 - xF'_P(F_\pi(x))}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
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- Our result for standard random networks:
  \[ \langle n \rangle = F'_\pi(1) = \frac{1 - S_1}{1 - \langle k \rangle (1 - S_1)} \]

  - Recall that \( \langle k \rangle = 1 \) is the critical value of average degree for standard random networks.
  - Look at what happens when we increase \( \langle k \rangle \) to 1 from below.
  - We have \( S_1 = 0 \) for all \( \langle k \rangle < 1 \) so
    \[ \langle n \rangle = \frac{1}{1 - \langle k \rangle} \]

  - This blows up as \( \langle k \rangle \to 1 \).
  - **Reason:** we have a power law distribution of component sizes at \( \langle k \rangle = 1 \).
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- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}$.
- For $\langle k \rangle < 1$, $S_1 \sim \log N$. 
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