Random walks and diffusion on networks

Complex Networks, Course 303A, Spring, 2009

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Outline

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References
Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node $j$ and take time to be discrete.
- Q: What’s the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\bar{p}(t)$.
- First task: connect $\bar{p}(t + 1)$ to $\bar{p}(t)$. 

Let’s call our walker Barry. Unfortunately for Barry, he lives on a high dimensional graph and is far from home. Worse still: Barry is hopelessly drunk.
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- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\tilde{p}(t)$.
- First task: connect $\tilde{p}(t + 1)$ to $\tilde{p}(t)$.
- Let’s call our walker Barry.
Random walks on networks—basics:

▶ Imagine a single random walker moving around on a network.

▶ At \( t = 0 \), start walker at node \( j \) and take time to be discrete.

▶ **Q:** What’s the long term probability distribution for where the walker will be?

▶ Define \( p_i(t) \) as the probability that at time step \( t \), our walker is at node \( i \).

▶ We want to characterize the evolution of \( \vec{p}(t) \).

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Where is Barry?

Consider simple undirected networks with an edge either present or absent.

Represent network by a symmetric adjacency matrix $A$ where

\[ a_{ij} = 1 \text{ if } i \text{ and } j \text{ are connected}, \]
\[ a_{ij} = 0 \text{ otherwise}. \]

Barry is at node $i$ at time $t$ with probability $p_i(t)$.

In the next time step he randomly lurches toward one of $i$’s neighbors.

Equation-wise:

\[ p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{k_i} a_{ji} p_i(t). \]

where $k_i$ is $i$’s degree.
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- Consider simple undirected networks with an edges either present of absent.
- Represent network by a symmetric **adjacency matrix** $A$ where
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  a_{ij} = \begin{cases} 
  1 & \text{if } i \text{ and } j \text{ are connected,} \\
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  \[ p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{k_i} a_{ji} p_i(t). \]
  where $k_i$ is $i$'s degree. Note: $k_i = \sum_{j=1}^{n} a_{ij}$. 
Where is Barry?

- Linear algebra-based excitement:
  \[ p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{k_i} a_{ji} p_i(t) \]
  is more usefully viewed as
  \[ \vec{p}(t + 1) = A K^{-1} \vec{p}(t) \]
  where \([K_{ij}] = [\delta_{ij} k_i]\) has node degrees on the main diagonal and zeros everywhere else.

- So... we need to find the dominant eigenvalue of \(A K^{-1}\).

- Expect this eigenvalue will be 1 (doesn’t make sense for total probability to change).

- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).

- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.
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- By inspection, we see that
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  \vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k}
  \]
  satisfies \(\vec{p}(\infty) = AK^{-1} \vec{p}(\infty)\) with eigenvalue 1.

- We will find Barry at node \(i\) with probability proportional to its degree \(k_i\).

- Nice implication: probability of finding Barry travelling along any edge is **uniform**.

- Diffusion in real space smooths things out.

- On networks, uniformity occurs on edges.

- So in fact, diffusion in real space is about the edges too but we just don’t see that.
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Other pieces:

- **Good news:** $AK^{-1}$ is similar to a real symmetric matrix.
- Consider the transformation $M = K^{-1/2}$:

  $$K^{-1/2}AK^{-1}K^{1/2} = K^{-1/2}AK^{-1/2}.$$ 

- Since $A^T = A$, we have

  $$(K^{-1/2}AK^{-1/2})^T = K^{-1/2}AK^{-1/2}.$$ 

- **Upshot:** $AK^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.
- Other goodies: next time round.
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