Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node $j$ and take time to be discrete.
- Q: What’s the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\bar{p}(t)$.
- First task: connect $\bar{p}(t + 1)$ to $\bar{p}(t)$.
- Let’s call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is hopelessly drunk.

Where is Barry?

- Consider simple undirected networks with an edges either present of absent.
- Represent network by a symmetric adjacency matrix $A$ where
  
  $a_{ij} = 1$ if $i$ and $j$ are connected,
  
  $a_{ij} = 0$ otherwise.

- Barry is at node $i$ at time $t$ with probability $p_i(t)$.
- In the next time step he randomly lurches toward one of $i$’s neighbors.
- Equation-wise:
  
  $p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{k_i} a_{ij} p_i(t)$.

  where $k_i$ is $i$’s degree. Note: $k_i = \sum_{j=1}^{n} a_{ij}$. 
Where is Barry?

- Linear algebra-based excitement:
  \[ p_j(t + 1) = \sum_{i=1}^{n} \frac{1}{K_i} a_{ij} p_i(t) \]
  is more usefully viewed as
  \[ \bar{p}(t + 1) = AK^{-1} \bar{p}(t) \]
  where \([K_i] = [\delta_{ij}k_i]\) has node degrees on the main diagonal and zeros everywhere else.
- So... we need to find the dominant eigenvalue of 
  \(AK^{-1}\).
- Expect this eigenvalue will be 1 (doesn’t make sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.

Other pieces:

- Good news: \(AK^{-1}\) is similar to a real symmetric matrix.
- Consider the transformation \(M = K^{-1/2}\):
  \[ K^{-1/2}AK^{-1} K^{1/2} = K^{-1/2} AK^{-1/2}. \]
- Since \(A^T = A\), we have
  \[ (K^{-1/2}AK^{-1/2})^T = K^{-1/2} AK^{-1/2}. \]
- Upshot: \(AK^{-1}\) has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.
- Other goodies: next time round.

Where is Barry?

- By inspection, we see that
  \[ \bar{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \bar{k} \]
  satisfies \(\bar{p}(\infty) = AK^{-1} \bar{p}(\infty)\) with eigenvalue 1.
- We will find Barry at node \(i\) with probability proportional to its degree \(k_i\).
- Nice implication: probability of finding Barry travelling along any edge is uniform.
- Diffusion in real space smooths things out.
- On networks, uniformity occurs on edges.
- So in fact, diffusion in real space is about the edges too but we just don’t see that.

References:

- ▶ Good news: \(AK^{-1}\) is similar to a real symmetric matrix.
- ▶ Consider the transformation \(M = K^{-1/2}\):
  \[ K^{-1/2}AK^{-1} K^{1/2} = K^{-1/2} AK^{-1/2}. \]
- ▶ Since \(A^T = A\), we have
  \[ (K^{-1/2}AK^{-1/2})^T = K^{-1/2} AK^{-1/2}. \]
- ▶ Upshot: \(AK^{-1}\) has real eigenvalues and a complete set of orthogonal eigenvectors.
- ▶ Can also show that maximum eigenvalue magnitude is indeed 1.
- ▶ Other goodies: next time round.