Branching Networks II
Complex Networks, Course 303A, Spring, 2009

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.
Outline

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models

Nutshell

References
Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- In terms of network architecture, Horton’s laws appear to contain less detailed information than Tokunaga’s law.
- Oddly, Horton’s laws have **four** parameters and Tokunaga has **two** parameters.
- $R_n$, $R_a$, $R_\ell$, and $R_s$ versus $T_1$ and $R_T$. One simple redundancy: $R_\ell = R_s$.
  Insert question from assignment 1
- To make a connection, clearest approach is to start with Tokunaga’s law...
- Known result: Tokunaga $\rightarrow$ Horton $^{[18, 19, 20, 9, 2]}$
Let us make them happy

We need one more ingredient:

Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:
  Drainage density $\rho_{dd} = \frac{1}{\text{inverse of typical distance between channels in a landscape}}$
- In terms of basin characteristics:

$$\rho_{dd} \sim \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \sum_{\omega=1}^{\Omega} \frac{n_\omega \bar{s}_\omega}{a_\Omega}$$
More with the happy-making thing

Start with Tokunaga’s law: \( T_k = T_1 R_T^{k-1} \)

- Start looking for Horton’s stream number law:
  \[ \frac{n_\omega}{n_{\omega+1}} = R_n. \]
- Estimate \( n_\omega \), the number of streams of order \( \omega \) in terms of other \( n_{\omega'}, \omega' > \omega \).
- Observe that each stream of order \( \omega \) terminates by either:

1. Running into another stream of order \( \omega \) and generating a stream of order \( \omega + 1 \)...  
   - \( 2n_{\omega+1} \) streams of order \( \omega \) do this

2. Running into and being absorbed by a stream of higher order \( \omega' > \omega \)...  
   - \( n'_\omega T_{\omega'-\omega} \) streams of order \( \omega \) do this
More with the happy-making thing

Putting things together:

\[ n_\omega = \left(2n_{\omega+1} + \sum_{\omega' = \omega+1}^{\Omega} T_{\omega'} - \omega n_{\omega'} \right) \]

- Use Tokunaga’s law and manipulate expression to create \( R_n \)'s.
- Insert question from assignment 1

Solution:

\[ R_n = \left(2 + R_T + T_1\right) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T} \]

(The larger value is the one we want.)
Finding other Horton ratios

Connect Tokunaga to $R_s$

- Now use uniform drainage density $\rho_{dd}$.
- Assume side streams are roughly separated by distance $1/\rho_{dd}$.
- For an order $\omega$ stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + \sum_{k=1}^{\omega-1} T_k \right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left( 1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right) \propto R_T^\omega$$
Horton and Tokunaga are happy

Altogether then:

▶

\[ \frac{s_\omega}{s_{\omega-1}} = R_T \Rightarrow R_s = R_T \]

▶ Recall \( R_\ell = R_s \) so

\[ R_\ell = R_T \]

▶ And from before:

\[ R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2} \]
Horton and Tokunaga are happy

Some observations:

- $R_n$ and $R_\ell$ depend on $T_1$ and $R_T$.
- Seems that $R_a$ must as well...
- Suggests Horton’s laws must contain some redundancy
- We’ll in fact see that $R_a = R_n$.
- Also: Both Tokunaga’s law and Horton’s laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]
Horton and Tokunaga are happy

The other way round

- Note: We can invert the expressions for $R_n$ and $R_\ell$ to find Tokunaga’s parameters in terms of Horton’s parameters.

- \[ R_T = R_\ell, \]

- \[ T_1 = R_n - R_\ell - 2 + 2R_\ell / R_n. \]

- Suggests we should be able to argue that Horton’s laws imply Tokunaga’s laws (if drainage density is uniform)…
Horton and Tokunaga are friends

From Horton to Tokunaga \(^{[2]}\)

- Assume Horton's laws hold for number and length.
- Start with an order \(\omega\) stream.
- Scale up by a factor of \(R_\ell\), orders increment.
- Maintain drainage density by adding new order 1 streams.
Horton and Tokunaga are friends

...and in detail:

- Must retain same drainage density.
- Add an extra \((R_\ell - 1)\) first order streams for each original tributary.
- Since number of first order streams is now given by \(T_{k+1}\) we have:

\[
T_{k+1} = (R_\ell - 1) \left( \sum_{i=1}^{k} T_i + 1 \right).
\]

- For large \(\omega\), Tokunaga’s law is the solution—let’s check...
Horton and Tokunaga are friends

Just checking:

- Substitute Tokunaga’s law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

$$T_{k+1} = (R_\ell - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

- $$T_{k+1} = (R_\ell - 1) \left( \sum_{i=1}^{k} T_1 R_\ell^{i-1} + 1 \right)$$

$$= (R_\ell - 1) T_1 \left( \frac{R_\ell^k - 1}{R_\ell - 1} + 1 \right)$$

$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^k}{R_\ell - 1} = T_1 R_\ell^k \quad \ldots \text{yep.}$$
Horton’s laws of area and number:

- In right plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$...
Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.
Mississippi:

<table>
<thead>
<tr>
<th>ω range</th>
<th>$R_n$</th>
<th>$R_a$</th>
<th>$R_\ell$</th>
<th>$R_s$</th>
<th>$R_a/R_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2, 3]</td>
<td>5.27</td>
<td>5.26</td>
<td>2.48</td>
<td>2.30</td>
<td>1.00</td>
</tr>
<tr>
<td>[2, 5]</td>
<td>4.86</td>
<td>4.96</td>
<td>2.42</td>
<td>2.31</td>
<td>1.02</td>
</tr>
<tr>
<td>[2, 7]</td>
<td>4.77</td>
<td>4.88</td>
<td>2.40</td>
<td>2.31</td>
<td>1.02</td>
</tr>
<tr>
<td>[3, 4]</td>
<td>4.72</td>
<td>4.91</td>
<td>2.41</td>
<td>2.34</td>
<td>1.04</td>
</tr>
<tr>
<td>[3, 6]</td>
<td>4.70</td>
<td>4.83</td>
<td>2.40</td>
<td>2.35</td>
<td>1.03</td>
</tr>
<tr>
<td>[3, 8]</td>
<td>4.60</td>
<td>4.79</td>
<td>2.38</td>
<td>2.34</td>
<td>1.04</td>
</tr>
<tr>
<td>[4, 6]</td>
<td>4.69</td>
<td>4.81</td>
<td>2.40</td>
<td>2.36</td>
<td>1.02</td>
</tr>
<tr>
<td>[4, 8]</td>
<td>4.57</td>
<td>4.77</td>
<td>2.38</td>
<td>2.34</td>
<td>1.05</td>
</tr>
<tr>
<td>[5, 7]</td>
<td>4.68</td>
<td>4.83</td>
<td>2.36</td>
<td>2.29</td>
<td>1.03</td>
</tr>
<tr>
<td>[6, 7]</td>
<td>4.63</td>
<td>4.76</td>
<td>2.30</td>
<td>2.16</td>
<td>1.03</td>
</tr>
<tr>
<td>[7, 8]</td>
<td>4.16</td>
<td>4.67</td>
<td>2.41</td>
<td>2.56</td>
<td>1.12</td>
</tr>
<tr>
<td>mean $\mu$</td>
<td>4.69</td>
<td>4.85</td>
<td>2.40</td>
<td>2.33</td>
<td>1.04</td>
</tr>
<tr>
<td>std dev $\sigma$</td>
<td>0.21</td>
<td>0.13</td>
<td>0.04</td>
<td>0.07</td>
<td>0.03</td>
</tr>
<tr>
<td>$\sigma/\mu$</td>
<td>0.045</td>
<td>0.027</td>
<td>0.015</td>
<td>0.031</td>
<td>0.024</td>
</tr>
</tbody>
</table>
### Amazon:

<table>
<thead>
<tr>
<th>$\omega$ range</th>
<th>$R_n$</th>
<th>$R_a$</th>
<th>$R_\ell$</th>
<th>$R_s$</th>
<th>$R_a/R_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[2, 3]$</td>
<td>4.78</td>
<td>4.71</td>
<td>2.47</td>
<td>2.08</td>
<td>0.99</td>
</tr>
<tr>
<td>$[2, 5]$</td>
<td>4.55</td>
<td>4.58</td>
<td>2.32</td>
<td>2.12</td>
<td>1.01</td>
</tr>
<tr>
<td>$[2, 7]$</td>
<td>4.42</td>
<td>4.53</td>
<td>2.24</td>
<td>2.10</td>
<td>1.02</td>
</tr>
<tr>
<td>$[3, 5]$</td>
<td>4.45</td>
<td>4.52</td>
<td>2.26</td>
<td>2.14</td>
<td>1.01</td>
</tr>
<tr>
<td>$[3, 7]$</td>
<td>4.35</td>
<td>4.49</td>
<td>2.20</td>
<td>2.10</td>
<td>1.03</td>
</tr>
<tr>
<td>$[4, 6]$</td>
<td>4.38</td>
<td>4.54</td>
<td>2.22</td>
<td>2.18</td>
<td>1.03</td>
</tr>
<tr>
<td>$[5, 6]$</td>
<td>4.38</td>
<td>4.62</td>
<td>2.22</td>
<td>2.21</td>
<td>1.06</td>
</tr>
<tr>
<td>$[6, 7]$</td>
<td>4.08</td>
<td>4.27</td>
<td>2.05</td>
<td>1.83</td>
<td>1.05</td>
</tr>
<tr>
<td>mean $\mu$</td>
<td>4.42</td>
<td>4.53</td>
<td>2.25</td>
<td>2.10</td>
<td>1.02</td>
</tr>
<tr>
<td>std dev $\sigma$</td>
<td>0.17</td>
<td>0.10</td>
<td>0.10</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>$\sigma/\mu$</td>
<td>0.038</td>
<td>0.023</td>
<td>0.045</td>
<td>0.042</td>
<td>0.019</td>
</tr>
</tbody>
</table>
Reducing Horton’s laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto$ sum of all stream lengths in a order $\Omega$ basin (assuming uniform drainage density)

- So:

\[
a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{dd}
\]

\[
\propto \sum_{\omega=1}^{\Omega} R_n^{\Omega-\omega} \cdot \frac{n_\omega}{n_\omega} \cdot \bar{s}_1 \cdot R_s^{\omega-1}
\]

\[
= \frac{R_n}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^\omega
\]
Reducing Horton’s laws:

Continued ...

\[ a_\Omega \propto \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^\omega \]

\[ = \frac{R_n^\Omega}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^\Omega}{1 - (R_s/R_n)} \]

\[ \sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1 - (R_s/R_n)} \text{ as } \Omega \uparrow \]

So, \( a_\Omega \) is growing like \( R_n^\Omega \) and therefore:

\[ R_n \equiv R_a \]
Reducing Horton’s laws:

Not quite:

- … But this only a rough argument as Horton’s laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Insert question from assignment 1 (ichi)
Equipartitioning:

Intriguing division of area:

- Observe: Combined area of basins of order $\omega$ independent of $\omega$.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:
  \[ R_n \equiv R_a \Rightarrow n_\omega \bar{a}_\omega = \text{const} \]
- Reason:
  \[ n_\omega \propto (R_n)^{-\omega} \]
  \[ \bar{a}_\omega \propto (R_a)^{\omega} \propto n_\omega^{-1} \]
Equipartitioning:
Some examples:
Scaling laws

The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga’s law describes detailed architecture: 
  \[ T_k = T_1 R_T^{k-1}. \]
- We have connected Tokunaga’s and Horton’s laws
- Only two Horton laws are independent \((R_n = R_a)\)
- Only two parameters are independent: 
  \((T_1, R_T) \iff (R_n, R_s)\)
Scaling laws

A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network \( p \).
- Each point \( p \) is associated with a basin and a longest stream length
  - Q: What is probability that the \( p \)'s drainage basin has area \( a \)? \( P(a) \propto a^{-\tau} \) for large \( a \)
  - Q: What is probability that the longest stream from \( p \) has length \( \ell \)? \( P(\ell) \propto \ell^{-\gamma} \) for large \( \ell \)
  - Roughly observed: \( 1.3 \lesssim \tau \lesssim 1.5 \) and \( 1.7 \lesssim \gamma \lesssim 2.0 \)
Scaling laws

Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf’s law)
  - Word frequency (Zipf’s law) [21]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]

- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...
- Our task is always to illuminate the mechanism...
Scaling laws

Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story \cite{17, 1, 2}
- Let’s work on $P(\ell)$...
- Our first fudge: assume Horton’s laws hold throughout a basin of order $\Omega$.
- (We know they deviate from strict laws for low $\omega$ and high $\omega$ but not too much.)
Scaling laws

Finding $\gamma$:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_>(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\text{max}}} P(\ell) d\ell$$

- Also known as the exceedance probability.

$$P_>(\ell_*) = 1 - P(\ell < \ell_*)$$
Scaling laws

Finding $\gamma$:

- The connection between $P(x)$ and $P_>(x)$ when $P(x)$ has a power law tail is simple:
- Given $P(\ell) \sim \ell^{-\gamma}$ large $\ell$ then for large enough $\ell_*$

\[
P_>(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\text{max}}} P(\ell) \, d\ell
\]

\[
\sim \int_{\ell=\ell_*}^{\ell_{\text{max}}} \ell^{-\gamma} \, d\ell
\]

\[
= \frac{\ell^{-\gamma+1}}{-\gamma+1} \bigg|_{\ell=\ell_*}^{\ell_{\text{max}}}
\]

\[
\propto \ell_*^{-\gamma+1} \quad \text{for } \ell_{\text{max}} \gg \ell_*
\]
Scaling laws

Finding $\gamma$:

- **Aim**: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution $\Delta$
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
- Approximate $P_{>}(\ell_*)$ as

\[ P_{>}(\ell_*) = \frac{N_{>}(\ell_*; \Delta)}{N_{>}(0; \Delta)}. \]

where $N_{>}(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$. 

- Use Horton’s law of stream segments:

\[ s_\omega / s_{\omega-1} = R_s \ldots \]
Scaling laws

Finding \( \gamma \):

- Set \( \ell_* = \ell_\omega \) for some \( 1 \ll \omega \ll \Omega \).

\[
P_>(\ell_\omega) = \frac{N_>(\ell_\omega; \Delta)}{N_>(0; \Delta)} \approx \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'} / \Delta}
\]

- \( \Delta \)'s cancel

- Denominator is \( a_\Omega \rho_{dd} \), a constant.

- So... using Horton's laws...

\[
P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \approx \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})
\]
Scaling laws

Finding $\gamma$:

- We are here:

$$P_>(\ell_\omega) \propto \sum_{\omega' = \omega + 1}^{\Omega} (1 \cdot R_n^{\Omega - \omega'})(\bar{s}_1 \cdot R_s^{\omega' - 1})$$

- Cleaning up irrelevant constants:

$$P_>(\ell_\omega) \propto \sum_{\omega' = \omega + 1}^{\Omega} \left( \frac{R_s}{R_n} \right)^{\omega'}$$

- Change summation order by substituting $\omega'' = \Omega - \omega'$.

- Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)
Scaling laws

Finding $\gamma$:

$P > (\ell_\omega) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left( \frac{R_s}{R_n} \right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left( \frac{R_n}{R_s} \right)^{\omega''}$

Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$,

$P > (\ell_\omega) \propto \left( \frac{R_n}{R_s} \right)^{\Omega-\omega} \propto \left( \frac{R_n}{R_s} \right)^{-\omega}$

again using $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$
Scaling laws

Finding $\gamma$:

- Nearly there:

$$P_>(\ell_\omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

- Need to express right hand side in terms of $\ell_\omega$.
- Recall that $\ell_\omega \sim \bar{\ell}_1 R_{\ell}^{-1}$.

$$\ell_\omega \propto R_{\ell}^\omega = R_s^\omega = e^{\omega \ln R_s}$$
Scaling laws

Finding $\gamma$:

- Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}\right)^{-\ln(R_n/R_s)/\ln(R_s)}$$

- $$\propto \ell_\omega - \ln(R_n/R_s)/\ln R_s$$

- $$= \ell_\omega - (\ln R_n - \ln R_s)/\ln R_s$$

- $$= \ell_\omega - \ln R_n/\ln R_s + 1$$

- $$= \ell_\omega^{\gamma + 1}$$
Scaling laws

Finding $\gamma$:

- And so we have:

\[ \gamma = \ln \frac{R_n}{\ln R_s} \]

- Proceeding in a similar fashion, we can show:

\[ \tau = 2 - \ln \frac{R_s}{\ln R_n} = 2 - \frac{1}{\gamma} \]

Insert question from assignment 1

- Such connections between exponents are called scaling relations

- Let's connect to one last relationship: Hack's law
Scaling laws

Hack’s law: \[^{[6]}\]

\[ \ell \propto a^h \]

- Typically observed that \(0.5 \lesssim h \lesssim 0.7\).
- Use Horton laws to connect \(h\) to Horton ratios:

\[ \ell_\omega \propto R_s^\omega \quad \text{and} \quad a_\omega \propto R_n^\omega \]

- Observe:

\[ \ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}\right)^{\ln R_s / \ln R_n} \]

\[ \propto \left(R_n^\omega\right)^{\ln R_s / \ln R_n} \propto a_\omega^{\ln R_s / \ln R_n} \quad \Rightarrow \quad h = \ln R_s / \ln R_n \]
Connecting exponents
Only 3 parameters are independent: e.g., take $d$, $R_n$, and $R_s$

<table>
<thead>
<tr>
<th>relation:</th>
<th>scaling relation/parameter: $[2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell \sim L^d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$T_k = T_1(R_T)^{k-1}$</td>
<td>$T_1 = R_n - R_s - 2 + 2R_s/R_n$</td>
</tr>
<tr>
<td>$n_\omega/n_{\omega+1} = R_n$</td>
<td>$R_n$</td>
</tr>
<tr>
<td>$\bar{a}<em>{\omega+1}/\bar{a}</em>\omega = R_a$</td>
<td>$R_a = R_n$</td>
</tr>
<tr>
<td>$\bar{\ell}<em>{\omega+1}/\bar{\ell}</em>\omega = R_\ell$</td>
<td>$R_\ell = R_s$</td>
</tr>
<tr>
<td>$\ell \sim a^h$</td>
<td>$h = \log R_s / \log R_n$</td>
</tr>
<tr>
<td>$a \sim L^D$</td>
<td>$D = d/h$</td>
</tr>
<tr>
<td>$L_\perp \sim L^H$</td>
<td>$H = d/h - 1$</td>
</tr>
<tr>
<td>$P(a) \sim a^{-\tau}$</td>
<td>$\tau = 2 - h$</td>
</tr>
<tr>
<td>$P(\ell) \sim \ell^{-\gamma}$</td>
<td>$\gamma = 1/h$</td>
</tr>
<tr>
<td>$\Lambda \sim a^\beta$</td>
<td>$\beta = 1 + h$</td>
</tr>
<tr>
<td>$\lambda \sim L^\varphi$</td>
<td>$\varphi = d$</td>
</tr>
</tbody>
</table>
Equipartitioning reexamined:

Recall this story:

Mississippi basin partitioning

Amazon basin partitioning

Nile basin partitioning
What about $P(a) \sim a^{-\tau}$?

Since $\tau > 1$, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

$P(a)$ overcounts basins within basins...

while stream ordering separates basins...
Fluctuations

Moving beyond the mean:

- Both Horton’s laws and Tokunaga’s law relate average properties, e.g.,

  \[ \bar{s}_\omega / \bar{s}_{\omega-1} = R_s \]

- Natural generalization to consideration relationships between probability distributions

- Yields rich and full description of branching network structure

- See into the heart of randomness...
A toy model—Scheidegger’s model

Directed random networks $^{[11, 12]}$

- $P(\downarrow) = P(\leftarrow) = 1/2$
- Flow is directed downwards
- Useful and interesting test case—more later...
Generalizing Horton’s laws

- \( \bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell | \omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell / R_\ell^\omega) \)
- \( \bar{a}_\omega \propto (R_a)^\omega \Rightarrow N(a | \omega) = (R_n^2)^{-\omega} F_a(a / R_n^\omega) \)

Mississippi: length distributions

- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order
Generalizing Horton’s laws

- How well does overall basin fit internal pattern?

- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44%
- Okay.
Generalizing Horton’s laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in $10^3$ km):

<table>
<thead>
<tr>
<th>basin:</th>
<th>$l_\Omega$</th>
<th>$\bar{l}_\Omega$</th>
<th>$\sigma_l$</th>
<th>$l/\bar{l}_\Omega$</th>
<th>$\sigma_l/\bar{l}_\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mississippi</td>
<td>4.92</td>
<td>11.10</td>
<td>5.60</td>
<td>0.44</td>
<td>0.51</td>
</tr>
<tr>
<td>Amazon</td>
<td>5.75</td>
<td>9.18</td>
<td>6.85</td>
<td>0.63</td>
<td>0.75</td>
</tr>
<tr>
<td>Nile</td>
<td>6.49</td>
<td>2.66</td>
<td>2.20</td>
<td>2.44</td>
<td>0.83</td>
</tr>
<tr>
<td>Congo</td>
<td>5.07</td>
<td>10.13</td>
<td>5.75</td>
<td>0.50</td>
<td>0.57</td>
</tr>
<tr>
<td>Kansas</td>
<td>1.07</td>
<td>2.37</td>
<td>1.74</td>
<td>0.45</td>
<td>0.73</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>basin:</th>
<th>$a$</th>
<th>$\bar{a}_\Omega$</th>
<th>$\sigma_a$</th>
<th>$a/\bar{a}_\Omega$</th>
<th>$\sigma_a/\bar{a}_\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mississippi</td>
<td>2.74</td>
<td>7.55</td>
<td>5.58</td>
<td>0.36</td>
<td>0.74</td>
</tr>
<tr>
<td>Amazon</td>
<td>5.40</td>
<td>9.07</td>
<td>8.04</td>
<td>0.60</td>
<td>0.89</td>
</tr>
<tr>
<td>Nile</td>
<td>3.08</td>
<td>0.96</td>
<td>0.79</td>
<td>3.19</td>
<td>0.82</td>
</tr>
<tr>
<td>Congo</td>
<td>3.70</td>
<td>10.09</td>
<td>8.28</td>
<td>0.37</td>
<td>0.82</td>
</tr>
<tr>
<td>Kansas</td>
<td>0.14</td>
<td>0.49</td>
<td>0.42</td>
<td>0.28</td>
<td>0.86</td>
</tr>
</tbody>
</table>
Combining stream segments distributions:

- Stream segments sum to give main stream lengths

\[ \ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu \]

- \( P(\ell_\omega) \) is a convolution of distributions for the \( s_\omega \)
Generalizing Horton’s laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) \ast N(s|2) \ast \cdots \ast N(s|\omega)$$

Mississippi: $\xi \simeq 900$ m.
Generalizing Horton’s laws

▶ Next level up: Main stream length distributions must combine to give overall distribution for stream length

Mississippi: length distributions

\[ P(\ell) \sim \ell^{-\gamma} \]

▶ Another round of convolutions \([3]\)

▶ Interesting...
Generalizing Horton’s laws

Number and area distributions for the Scheidegger model $P(n_{1,6})$ versus $P(a_6)$. 

![Graph showing number and area distributions](image.png)
Generalizing Tokunaga’s law

Scheidegger:

- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using $R_s$
Generalizing Tokunaga’s law

Mississippi:

- Same data collapse for Mississippi...
Generalizing Tokunaga’s law

So

\[ P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[ \frac{T_{\mu,\nu}}{(R_s)^{\mu-\nu-1}} \right] \]

where

\[ P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}. \]

\[ P(s_{\mu}) \leftrightarrow P(T_{\mu,\nu}) \]

- Exponentials arise from randomness.
- Look at joint probability \( P(s_{\mu}, T_{\mu,\nu}) \).
Generalizing Tokunaga’s law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments
Generalizing Tokunaga’s law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order $\mu$ stream segment terminating is constant:
  \[ \tilde{p}_\mu \simeq \frac{1}{(R_s)^{\mu-1}} \xi_s \]
- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- $\Rightarrow$ random spatial distribution of stream segments
Generalizing Tokunaga’s law

- Joint distribution for generalized version of Tokunaga’s law:

\[ P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \left( \frac{s_\mu - 1}{T_{\mu,\nu}} \right) p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu) s_\mu - T_{\mu,\nu} - 1 \]

where

- \( p_\nu \) = probability of absorbing an order \( \nu \) side stream
- \( \tilde{p}_\mu \) = probability of an order \( \mu \) stream terminating

- Approximation: depends on distance units of \( s_\mu \)

- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.
Generalizing Tokunaga’s law

- Now deal with thing:

\[ P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \left( \frac{s_\mu - 1}{T_{\mu,\nu}} \right) p_{\nu}^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu) s_\mu - T_{\mu,\nu} - 1 \]

- Set \((x, y) = (s_\mu, T_{\mu,\nu})\) and \(q = 1 - p_\nu - \tilde{p}_\mu\), approximate liberally.

- Obtain

\[ P(x, y) = N x^{-1/2} [F(y/x)]^x \]

where

\[ F(\nu) = \left( \frac{1 - \nu}{q} \right)^{(1-\nu)} \left( \frac{\nu}{p} \right)^{-\nu} \]
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:

\[ v = \frac{T_{\mu,\nu}}{l_\mu^{(s)}} \]

\[ P(v | l_\mu^{(s)}) \]

\[ [F(v)]_\mu^{(s)} \]
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Mississippi:

![Graphs showing the relationship between $T_{\mu,\nu}/l_\mu^{(s)}$ and $\log_{10} P(T_{\mu,\nu}/l_\mu^{(s)})$ for Mississippi data.](image)
Models

Random subnetworks on a Bethe lattice\textsuperscript{[13]}

- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of “Statistical inevitability” of river network statistics\textsuperscript{[7]}
- But Bethe lattices unconnected with surfaces.
- In fact, Bethe lattices $\sim$ infinite dimensional spaces (oops).
- So let’s move on...
Scheidegger’s model

Directed random networks \[11, 12\]

\[ P(\searrow) = P(\nearrow) = 1/2 \]

- Functional form of all scaling laws exhibited but exponents differ from real world \[15, 16, 14\]
A toy model—Scheidegger’s model

Random walk basins:

- Boundaries of basins are random walks

![Graph showing random walk basins with area a and axis labels n and x.](image-url)
Scheidegger’s model

Increasing partition of N=64

2
6
6
8 8 8 8
9 9
Scheidegger’s model

Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

\[ P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}. \]

and so \( P(\ell) \propto \ell^{-3/2}. \)

Typical area for a walk of length \( n \) is \( \propto n^{3/2}: \)

\[ \ell \propto a^{2/3}. \]

Find \( \tau = 4/3, \ h = 2/3, \ \gamma = 3/2, \ d = 1. \)

Note \( \tau = 2 - h \) and \( \gamma = 1/h. \)

\( R_n \) and \( R_\ell \) have not been derived analytically.
Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [10]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int d\vec{r} \ (\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^\gamma$$

- Landscapes obtained numerically give exponents near that of real networks.

- But: numerical method used matters.

- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [8]
Theoretical networks

Summary of universality classes:

<table>
<thead>
<tr>
<th>network</th>
<th>h</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-convergent flow</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Directed random</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>Undirected random</td>
<td>5/8</td>
<td>5/4</td>
</tr>
<tr>
<td>Self-similar</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>OCN’s (I)</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>OCN’s (II)</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>OCN’s (III)</td>
<td>3/5</td>
<td>1</td>
</tr>
<tr>
<td>Real rivers</td>
<td>0.5–0.7</td>
<td>1.0–1.2</td>
</tr>
</tbody>
</table>

\[ h \Rightarrow \ell \propto a^h \text{ (Hack’s law).}\]

\[ d \Rightarrow \ell \propto L^d \parallel \text{ (stream self-affinity).}\]
Nutshell

Branching networks II Key Points:

▶ Horton’s laws and Tokunaga law all fit together.
▶ nb. for 2-d networks, these laws are ‘planform’ laws and ignore slope.
▶ Abundant scaling relations can be derived.
▶ Can take \( R_n, R_\ell, \) and \( d \) as three independent parameters necessary to describe all 2-d branching networks.
▶ For scaling laws, only \( h = \ln R_\ell / \ln R_n \) and \( d \) are needed.
▶ Laws can be extended nicely to laws of distributions.
▶ Numerous models of branching network evolution exist: nothing rock solid yet.
References I


References II


Studies of longitudinal stream profiles in Virginia and Maryland. 

Statistical inevitability of Horton’s laws and the apparent randomness of stream channel networks. 
References III

Universality classes of optimal channel networks. 

New results for self-similar trees with applications to river networks. 

*Fractal River Basins: Chance and Self-Organization*. 
References IV


References VI


*Human Behaviour and the Principle of Least-Effort.* 
Addison-Wesley, Cambridge, MA, 1949.