Chapter 6: Lecture 25
Linear Algebra, Course 124C, Spring, 2009

Prof. Peter Dodds
Department of Mathematics & Statistics
University of Vermont

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Best solution $\tilde{x}_*$ when $\tilde{b} = \tilde{p} + \tilde{e}$:

Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a ‘best’ orthonormal basis
  - The $\hat{v}_i$ span $R^n$
  - We find the $\hat{v}_i$ as eigenvectors of $A^T A$.
  - The $\hat{u}_i$ span $R^m$
  - We find the $\hat{u}_i$ as eigenvectors of $A A^T$.

Happy bases

- $\{\hat{v}_1, \ldots, \hat{v}_r\}$ span Row space
- $\{\hat{v}_{r+1}, \ldots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \ldots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$ span Left Null space

All the way with $A\tilde{x} = \tilde{b}$:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^T$.

Where $\tilde{x}$ lives:

- Row space $C(A^T) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.
- $\dim C(A^T) + \dim N(A) = r + (n - r) = n$
- Orthogonality: $C(A^T) \bigotimes N(A) = R^n$

Where $\tilde{b}$ lives:

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^T) \subset R^m$.
- $\dim C(A) + \dim N(A^T) = r + (m - r) = m$
- Orthogonality: $C(A) \bigotimes N(A^T) = R^m$
**Fundamental Theorem of Linear Algebra**

**How \( A\tilde{x} \) works:**
- \( A = U\Sigma V^T \)
- \( A \) sends each \( \tilde{v}_i \in C(A^T) \) to its partner \( \tilde{u}_i \in C(A) \) with a stretch/shrink factor \( \sigma_i > 0 \).
- \( A \) is diagonal with respect to these bases and has positive entries (all \( \sigma_i > 0 \)).
- When viewed the right way, any \( A \) is a diagonal matrix \( \Sigma \).

**Image approximation (80x60)**

**Idea: use SVD to approximate images**
- Interpret elements of matrix \( A \) as color values of an image.
- Truncate series SVD representation of \( A \):
  \[
  A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T
  \]
  - Use fact that \( \sigma_1 > \sigma_2 > \ldots > \sigma_r > 0 \).
  - Rank \( r = \min(m, n) \).
  - Rank \( r \) = # of pixels on shortest side.
  - For color: approximate 3 matrices (RGB).