All the way with $A\vec{x} = \vec{b}$:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^T$.

Where $\vec{x}$ lives:

- Row space $C(A^T) \subset \mathbb{R}^n$.
- (Right) Nullspace $N(A) \subset \mathbb{R}^n$.
- $\dim C(A^T) + \dim N(A) = r + (n - r) = n$.
- Orthogonality: $C(A^T) \oplus N(A) = \mathbb{R}^n$.

Where $\vec{b}$ lives:

- Column space $C(A) \subset \mathbb{R}^m$.
- Left Nullspace $N(A^T) \subset \mathbb{R}^m$.
- $\dim C(A) + \dim N(A^T) = r + (m - r) = m$.
- Orthogonality: $C(A) \otimes N(A^T) = \mathbb{R}^m$. 

The fundamental theorem of linear algebra

Approximating matrices with SVD

The basic idea
Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:

- Row Space
  - $A\vec{x}_r = \vec{p}$
  - $d = r$
  - $\vec{x}_r$
  - Null Space
  - $A\vec{x}_n = \vec{0}$
  - $d = n - r$
  - $\vec{x}_n$

- Column Space
  - $A\vec{x}_* = \vec{p}$
  - $d = r$
  - $\vec{x}_*$
  - $\vec{b}$
  - Left Null Space
  - $d = m - r$
  - $\vec{0}$
  - $\vec{p}$
  - $\vec{e}$

Spaces:
- Row Space
- Column Space
- Left Null Space

Dimensions:
- $d = r$
- $d = n - r$
- $d = m - r$
Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a ‘best’ orthonormal basis
- The $\hat{v}_i$ span $R^n$
- We find the $\hat{v}_i$ as eigenvectors of $A^T A$.
- The $\hat{u}_i$ span $R^m$
- We find the $\hat{u}_i$ as eigenvectors of $AA^T$.

Happy bases

- $\{\hat{v}_1, \ldots, \hat{v}_r\}$ span Row space
- $\{\hat{v}_{r+1}, \ldots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \ldots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$ span Left Null space
Fundamental Theorem of Linear Algebra

How $A\vec{x}$ works:

- $A = U\Sigma V^T$
- $A$ sends each $\vec{v}_i \in C(A^T)$ to its partner $\vec{u}_i \in C(A)$ with a stretch/shrink factor $\sigma_i > 0$.
- $A$ is diagonal with respect to these bases and has positive entries (all $\sigma_i > 0$).
- When viewed the right way, any $A$ is a diagonal matrix $\Sigma$. 
Image approximation (80x60)

Idea: use SVD to approximate images

- Interpret elements of matrix A as color values of an image.
- Truncate series SVD representation of A:

\[ A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T \]

- Use fact that \( \sigma_1 > \sigma_2 > \ldots > \sigma_r > 0 \).
- Rank \( r = \min(m, n) \).
- Rank \( r = \# \) of pixels on shortest side.
- For color: approximate 3 matrices (RGB).