Chapter 2: Lecture 2
Linear Algebra, Course 124C, Spring, 2009

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Solving \( A\vec{x} = \vec{b} \):

- We (people + computers) solve systems of linear equations by a systematic method of Elimination followed by Back substitution.
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

\[
\begin{align*}
-x_1 + 3x_2 &= 1 \\
2x_1 + x_2 &= 5
\end{align*}
\]

Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.
2. Swap rows if needed to create an ‘upper triangular form’
   e.g.
   \[
   \begin{align*}
   x_2 &= 3 \\
   2x_1 - x_2 &= -1 \\
   \Rightarrow \quad 2x_1 - x_2 &= -1 \quad \Rightarrow \quad x_2 &= 3
   \end{align*}
   \]

Gaussian elimination:

Solve:

\[
\begin{align*}
2x_1 - 3x_2 &= 3 \\
4x_1 - 5x_2 + x_3 &= 7 \\
2x_1 - x_2 - 3x_3 &= 5
\end{align*}
\]
Gaussian elimination:

Summary:
Using row operations, we turned this problem:

\[
A\vec{x} = \vec{b} : \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}
\]

into this problem:

\[
U\vec{x} = \vec{d} : \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}
\]

and the latter is easy to solve using back substitution.

Gaussian elimination:

The one true method:

▶ We simplify \( A \) using elimination in the same way every time.

▶ Eliminate entries one column at a time, moving left to right, and down each column.

\[
\begin{align*}
X &+ X + X + X = X \\
1 \downarrow &+ X + X + X = X \\
2 \downarrow &+ 4 \downarrow + X + X = X \\
3 &\nearrow + 5 \rightarrow + 6 + X = X
\end{align*}
\]

Gaussian elimination:

Defn:
The entries along \( U \)'s main diagonal are the pivots of \( A \).
(The pivots are hidden—elimination finds them.)

Defn:
A matrix with only zeros below the main diagonal is called upper triangular. A matrix with only zeros above the main diagonal is called lower triangular. We get from \( A \) to \( U \) and the latter is always upper triangular.

Defn:
Singular means a system has no unique solution.
▶ It may have no solutions or infinitely many solutions.
▶ Singular = archaic way of saying 'messed up.'

Truth:
If at least one pivot is zero, the matrix will be singular.
(but the reverse is not necessarily true.)

Gaussian elimination:

▶ To eliminate entry in row \( i \) of \( j \)th column, subtract a multiple \( \ell_{ij} \) of the \( j \)th row from \( i \).

▶ For example:

\[
\begin{align*}
2x_1 &+ 3x_2 + -2x_3 + x_4 = 1 \\
x_1 &- 7x_2 + 3x_3 + x_4 = 1 \\
-x_1 &- 3x_2 - x_3 + 5x_4 = -2 \\
2x_1 &+ x_2 - 2x_3 + 2x_4 = 0
\end{align*}
\]

\[
\ell_{21} = 1/2, \ell_{31} = -1/2, \ell_{41} = ?.
\]

▶ Note: we cannot find \( \ell_{32} \) etc., until we are finished with row 1. Pivots are hidden!

▶ Note: the denominator of each \( \ell_{ij} \) multiplier is the pivot in the \( j \)th column.