Power Law Size Distributions

Principles of Complex Systems
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Outline

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  Introduction
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  Zipf’s law
  Wild vs. Mild
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The Don

Extreme deviations in test cricket

Don Bradman’s batting average = 166% next best.
Size distributions

The sizes of many systems’ elements appear to obey an inverse power-law size distribution:

\[ P(\text{size } = x) \sim c x^{-\gamma} \]

where \( x_{\text{min}} < x < x_{\text{max}} \) and \( \gamma > 1 \)

- Typically, \( 2 < \gamma < 3 \).
- \( x_{\text{min}} = \) lower cutoff
- \( x_{\text{max}} = \) upper cutoff
Usually, only the tail of the distribution obeys a power law:

\[ P(x) \sim c x^{-\gamma} \text{ as } x \to \infty. \]

Still use term ‘power law distribution’
Many systems have discrete sizes $k$:

- Word frequency
- Node degree (as we have seen): # hyperlinks, etc.
- Number of citations for articles, court decisions, etc.

$$P(k) \sim c k^{-\gamma}$$

where $k_{\text{min}} \leq k \leq k_{\text{max}}$
Power law size distributions are sometimes called **Pareto distributions** after Italian scholar Vilfredo Pareto.

- Pareto noted wealth in Italy was distributed unevenly (80–20 rule).
- Term used especially by economists
Size distributions

- Negative linear relationship in log-log space:

\[
\log P(x) = \log c - \gamma \log x
\]
Size distributions

Examples:

- Earthquake magnitude (Gutenberg Richter law): 
  \[ P(M) \propto M^{-3} \]
- Number of war deaths: 
  \[ P(d) \propto d^{-1.8} \]
- Sizes of forest fires
- Sizes of cities: 
  \[ P(n) \propto n^{-2.1} \]
- Number of links to and from websites
Size distributions

Examples:

- Number of citations to papers: $P(k) \propto k^{-3}$.
- Individual wealth (maybe): $P(W) \propto W^{-2}$.
- Distributions of tree trunk diameters: $P(d) \propto d^{-2}$.
- The gravitational force at a random point in the universe: $P(F) \propto F^{-5/2}$.
- Diameter of moon craters: $P(d) \propto d^{-3}$.
- Word frequency: e.g., $P(k) \propto k^{-2.2}$ (variable)

(Note: Exponents range in error; see M.E.J. Newman arxiv.org/cond-mat/0412004v3)
Power-law distributions are..

- often called ‘heavy-tailed’
- or said to have ‘fat tails’

Important!:

- Inverse power laws aren’t the only ones:
  - lognormals, stretched exponentials, ...
George Kingsley Zipf:

- We’ll study Zipf’s law in depth...
Zipfian rank-frequency plots

Zipf’s way:

- $s_i =$ the size of the $i$th ranked object.
- $i = 1$ corresponds to the largest size.
- $s_1$ could be the frequency of occurrence of the most common word in a text.
- Zipf’s observation:

$$s_i \propto i^{-\alpha}$$
Power law distributions

Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See “The Black Swan” by Nassim Taleb[1])
Turkeys...

From “The Black Swan”[1]
Taleb’s table [1]

Mediocristan/Extremistan

► Most typical member is mediocre/Most typical is either giant or tiny

► Winners get a small segment/Winner take almost all effects

► When you observe for a while, you know what’s going on/It takes a very long time to figure out what’s going on

► Prediction is easy/Prediction is hard

► History crawls/History makes jumps

► Tyranny of the collective/Tyranny of the accidental
Complementary Cumulative Distribution Function:

CCDF:

\[ P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x) \]

\[ = \int_{x'=x}^{\infty} P(x')dx' \]

\[ \propto \int_{x'=x}^{\infty} (x')^{-\gamma}dx' \]

\[ = \frac{1}{-\gamma + 1} (x')^{-\gamma + 1} \bigg|_{x'=x}^{\infty} \]

\[ \propto x^{-\gamma + 1} \]
Complementary Cumulative Distribution Function:

**CCDF:**

\[ P_{\geq}(x) \propto x^{-\gamma+1} \]

- Use when tail of \( P \) follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.
Complementary Cumulative Distribution Function:

- **Discrete variables:**

\[ P_{\geq}(k) = P(k' \geq k) \]

\[ = \sum_{k'=k}^{\infty} P(k) \]

\[ \propto k^{-\gamma+1} \]

- Use integrals to approximate sums.
Size distributions

Brown Corpus (1,015,945 words):

CCDF:

\[ N > n \]

Zipf:

\[ n_i \]

- The, of, and, to, a, ... = ‘objects’
- ‘Size’ = word frequency
- **Beep:** CCDF and Zipf plots are related...
Size distributions

Observe:

- \( NP_\geq(x) \) = the number of objects with size at least \( x \) where \( N \) = total number of objects.
- If an object has size \( x_i \), then \( NP_\geq(x_i) \) is its rank \( i \).
- So

\[
x_i \propto i^{-\alpha} = (NP_\geq(x_i))^{-\alpha}
\]

\[
\propto x_i^{(-\gamma+1)(-\alpha)}
\]

Since \( P_\geq(x) \sim x^{-\gamma+1} \),

\[
\alpha = \frac{1}{\gamma - 1}
\]

A rank distribution exponent of \( \alpha = 1 \) corresponds to a size distribution exponent \( \gamma = 2 \).
Details on the lack of scale:

Let’s find the mean:

\[ \langle x \rangle = \int_{x=x_{\text{min}}}^{x=x_{\text{max}}} xP(x)dx \]

\[ = c \int_{x=x_{\text{min}}}^{x=x_{\text{max}}} xx^{-\gamma}dx \]

\[ = \frac{c}{2-\gamma} \left( x_{\text{max}}^{2-\gamma} - x_{\text{min}}^{2-\gamma} \right). \]
The mean:

$$\langle x \rangle \sim \frac{c}{2-\gamma} \left( x_{\text{max}}^{2-\gamma} - x_{\text{min}}^{2-\gamma} \right).$$

- Mean blows up with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.
And in general...

Moments:

- All moments depend only on cutoffs.
- No internal scale dominates (even matters).
- Compare to a Gaussian, exponential, etc.
Moments

For many real size distributions:

\[ 2 < \gamma < 3 \]

- mean is finite (depends on lower cutoff)
- \( \sigma^2 \) = variance is ‘infinite’ (depends on upper cutoff)
- Width of distribution is ‘infinite’
Moments

Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:
  Mean average deviation (MAD) =
  \[ \langle |x - \langle x \rangle| \rangle \]
- MAD is unpleasant analytically...
How sample sizes grow...

Given $P(x) \sim cx^{-\gamma}$:

- We can show that after $n$ samples, we expect the largest sample to be
  
  $$x_1 \gtrsim n^{1/(\gamma - 1)}$$

- Sampling from a ‘mild’ distribution gives a much slower growth with $n$.

- e.g., for $P(x) = \lambda e^{-\lambda x}$, we find
  
  $$x_1 \gtrsim \frac{1}{\lambda} \ln n.$$
References

N. N. Taleb.
*The Black Swan.*

G. K. Zipf.
*Human Behaviour and the Principle of Least-Effort.*
Addison-Wesley, Cambridge, MA, 1949.