Chapter 2: Lecture 1
Linear Algebra, Course 124B, Fall, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont
Outline

Importance

Usages

Key problems

Three ways of looking...

Colbert on Equations

References
Basics:

- **Instructor:** Prof. Peter Dodds
- **Lecture room and meeting times:**
  111 Lafayette, Tuesday and Thursday, 2:00 pm to 3:15 pm
- **Office:** 203 Lord House, 16 Colchester Avenue
- **E-mail:** pdodds@uvm.edu
- **Course website:**
- **Textbook:** “Introduction to Linear Algebra” (3rd ed.) by Gilbert Strang; Wellesley-Cambridge Press.
Admin:

Paper products:

1. Outline
Paper products:

1. Outline

Office hours:

- 9:00 am to 10:30 am
  Tuesday and Thursday
  Rm 203, Math Building
Grading breakdown:

1. Assignments (40%)
   - Ten one-week assignments.
   - Lowest assignment score will be dropped.
   - The last assignment cannot be dropped!
   - Each assignment will have a random bonus point question which has nothing to do with linear algebra.
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   - Three 75 minutes tests distributed throughout the course, all of equal weighting.
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2. Midterm exams (35%)
   - Three 75 minutes tests distributed throughout the course, all of equal weighting.

3. Final exam (24%)
   - Three hours of pure happiness.
   - Tuesday, December 16th, 2008, 3:30 pm to 6:30 pm, in 111 Lafayette.
Grading breakdown:

1. **Homework (0%)**—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.

2. **General attendance (1%)**—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline. Contributing to examples of linear algebra in action for the class blog will help too.
How grading works:

Questions are worth 3 points according to the following scale:

- 3 = correct or very nearly so.
- 2 = acceptable but needs some revisions.
- 1 = needs major revisions.
- 0 = way off.
The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

<table>
<thead>
<tr>
<th>Week # (dates)</th>
<th>Tuesday</th>
<th>Thursday</th>
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<tbody>
<tr>
<td>1 (9/2, 9/4)</td>
<td>Lecture</td>
<td>Lecture ★ A1</td>
</tr>
<tr>
<td>2 (9/9, 9/11)</td>
<td>Lecture</td>
<td>Lecture ★ A2</td>
</tr>
<tr>
<td>3 (9/16, 9/18)</td>
<td>Lecture</td>
<td>Lecture ★ A3</td>
</tr>
<tr>
<td>4 (9/23, 9/25)</td>
<td>Lecture</td>
<td>Test 1</td>
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<td>5 (9/30, 10/2)</td>
<td>Lecture</td>
<td>Lecture ★ A4</td>
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<td>6 (10/7, 10/9)</td>
<td>Lecture</td>
<td>Lecture ★ A5</td>
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<td>7 (10/14, 10/16)</td>
<td>Lecture</td>
<td>Lecture ★ A6</td>
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<tr>
<td>8 (10/21, 10/23)</td>
<td>Lecture</td>
<td>Test 2</td>
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<tr>
<td>9 (10/28, 10/30)</td>
<td>Lecture</td>
<td>Lecture ★ A7</td>
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<tr>
<td>10 (11/4, 11/6)</td>
<td>Lecture</td>
<td>Lecture ★ A8</td>
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<td>11 (11/11, 11/13)</td>
<td>Lecture</td>
<td>Lecture ★ A9</td>
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<td>12 (11/18, 11/20)</td>
<td>Lecture</td>
<td>Test 3</td>
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<tr>
<td>13 (11/25, 11/27)</td>
<td>Thanksgiving</td>
<td>Thanksgiving</td>
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<tr>
<td>14 (12/2, 12/4)</td>
<td>Lecture</td>
<td>Lecture ★ A10</td>
</tr>
<tr>
<td>15 (12/9, 12/11)</td>
<td>Lecture</td>
<td>Review</td>
</tr>
</tbody>
</table>
Important dates:

1. Classes run from Tuesday, September 2nd to Thursday, December 11.
3. Last day to withdraw—Friday, October 31.
4. Reading and exam period—Friday, December 12th to Friday, December 19th.
More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.
More stuff:

Being good people:

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).
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2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.

3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See http://www.uvm.edu/cses/ for guidelines.
More stuff:

**Late policy:** Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

**Computing:** Students are encouraged to use Matlab or something similar to check their work.

**Note:** for assignment problems, written details of calculations will be required.
Grading:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Percentage</th>
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<tbody>
<tr>
<td>A+</td>
<td>97–100</td>
</tr>
<tr>
<td>A</td>
<td>93–96</td>
</tr>
<tr>
<td>A-</td>
<td>90–92</td>
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<tr>
<td>B+</td>
<td>87–89</td>
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<td>B</td>
<td>83–86</td>
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<td>B-</td>
<td>80–82</td>
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<td>C+</td>
<td>77–79</td>
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<td>C</td>
<td>73–76</td>
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<tr>
<td>C-</td>
<td>70–72</td>
</tr>
<tr>
<td>D+</td>
<td>67–69</td>
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<tr>
<td>D</td>
<td>63–66</td>
</tr>
<tr>
<td>D-</td>
<td>60–62</td>
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</tbody>
</table>
Why are we doing this?

Linear Algebra is

a body of mathematics

that deals with **discrete problems**.
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Many things are discrete:
- Information (0’s & 1’s, letters, words)
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- Sounds (musical notes)
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Many things are discrete:

- Information (0’s & 1’s, letters, words)
- People (sociology)
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- Sounds (musical notes)

Even more:
If real data is continuous, we almost always discretize it
(0’s and 1’s)
Why are we doing this?

Linear Algebra is used in many fields to solve problems:
- Engineering
- Computer Science (Google’s Pagerank)
- Physics
- Economics
- Biology
- Ecology
- …
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- Engineering
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- ...

Linear Algebra is as important as calculus.
Matrices as gadgets:

A transforms $\vec{x}$ into $\vec{x}'$ through multiplication

$$\vec{x}' = A\vec{x}$$

Can use matrices to:

- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things to different directions
Image approximation (80x60)

\[ A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{2} \sigma_i \hat{u}_i \hat{v}_i^T \]
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\[ A = \sum_{i=1}^{3} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Three key problems of Linear Algebra

1. Given a matrix $A$ and a vector $\vec{b}$, find $\vec{x}$ such that

$$A\vec{x} = \vec{b}.$$
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3. Coupled linear differential equations:
   \[ \frac{d}{dt} y(t) = A y(t) \]
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3. Coupled linear differential equations:
   \[ \frac{d}{dt}y(t) = A\, y(t) \]

- Our focus will be largely on #1, partly on #2.
Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:
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To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:

What is going on here? We have 26 lectures to find out...
Our friend $A\vec{x} = \vec{b}$

Broadly speaking, $A\vec{x} = \vec{b}$ translates as follows:

$\vec{b}$ represents reality (e.g., music, structure)
$A$ contains building blocks (e.g., notes, shapes)
$\vec{x}$ specifies how we combine our building blocks to represent $\vec{b}$.

How can we disentangle an orchestra's sound?
What about pictures, waves, signals, ...?
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- Compress information
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- See how we can alter information
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If we can represent reality as a superposition (or combination) of simple elements, we can do many things:

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- Compress information
- See how we can alter information
- Find a system’s simplest representation
- Find a system’s most important elements
- See how to adjust a system in a principled defined way
Three ways to understand $A\vec{x} = \vec{b}$:

- Way 1: The *Row* Picture
- Way 2: The *Column* Picture
- Way 3: The *Matrix* Picture

Example:

- $-x_1 + x_2 = 1$
- $2x_1 + x_2 = 4$

Call this a 2 by 2 system of equations.

2 equations with 2 unknowns.

Standard method of solving by adding and subtracting multiples of equations from each other.
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Row Picture—what we are doing:

Three ways of looking...
Three ways to understand $A\vec{x} = \vec{b}$:

Row Picture—what we are doing:

- (a) Finding intersection of two lines
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- A splendid and deep connection:
  (a) Geometry $\Leftrightarrow$ (b) Algebra
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Three possible kinds of solution:
Three ways to understand $A\vec{x} = \vec{b}$:

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Three possible kinds of solution:

1. Lines intersect at one point — One, unique solution
2. Lines are parallel and disjoint — No solutions
3. Lines are the same — Infinitely many solutions
Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:
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The column picture:

See

\[-x_1 + x_2 = 1 \]
\[2x_1 + x_2 = 4 \]
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**The column picture:**

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\[-x_1 + x_2 = 1\]
\[2x_1 + x_2 = 4\]

as

\[x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.\]
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x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.
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General problem

\[
x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}
\]

- Column vectors are ‘building blocks’
Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

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-x_1 + x_2 = 1 \\
2x_1 + x_2 = 4
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as

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$$

General problem

$$
x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}
$$

- Column vectors are ‘building blocks’
- **Key idea**: try to ‘reach’ $\vec{b}$ by combining multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$. 

References

Colbert on Equations
Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:
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Three possible kinds of solution:
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Three possible kinds of solution:

1. $\vec{a}_1 \parallel \vec{a}_2$: 1 solution
Three ways to understand $A\vec{x} = \vec{b}$:

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Three possible kinds of solution:

1. $\vec{a}_1 \parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: No solutions
Three ways to understand $A\vec{x} = \vec{b}$:

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Three possible kinds of solution:

1. $\vec{a}_1 \parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions
Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- Intuitive.
- Generalizes easily to many dimensions.

Three possible kinds of solution:

1. $\vec{a}_1 \parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions

Assuming neither $\vec{a}_1$ or $\vec{a}_1$ are $\vec{0}$. 
Three ways to understand $A\vec{x} = \vec{b}$:

**Difficulties:**

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
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**Difficulties:**

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
- **No!** We can still find the $\vec{x}$ that gets us as close to $\vec{b}$ as possible.
- Method of approximation—very important!
- We may not have the right building blocks but we can do our best.
Three ways to understand $A\vec{x} = \vec{b}$:

The Matrix Picture:
Three ways to understand $A\vec{x} = \vec{b}$:

The Matrix Picture:
Now see

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$
Three ways to understand $A\vec{x} = \vec{b}$:

**The Matrix Picture:**
Now see

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

as

$$A\vec{x} = \vec{b}: \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$
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The Matrix Picture:
Now see
\[
\begin{bmatrix}
-1 \\
2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
1 \\
1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 \\
4
\end{bmatrix}.
\]

as
\[
A\vec{x} = \vec{b} : \begin{bmatrix}
-1 & 1 \\
2 & 1
\end{bmatrix} \begin{bmatrix}
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\end{bmatrix} = \begin{bmatrix}
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4
\end{bmatrix}
\]

$A$ is now an operator:
- $A$ transforms $\vec{x}$ into $\vec{b}$. 
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$$A\vec{x} = \vec{b} : \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

$A$ is now an operator:

- $A$ transforms $\vec{x}$ into $\vec{b}$.
- In general, $A$ does two things to $\vec{x}$:
  1. Rotation
  2. Dilation (stretching/contraction)
The Matrix Picture

Key idea in linear algebra:

- Decomposition (or factorization) of matrices.
The Matrix Picture

Key idea in linear algebra:

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- Matrices can often be written as products or sums of simpler matrices
The Matrix Picture

Key idea in linear algebra:

- Decomposition (or factorization) of matrices.
- Matrices can often be written as products or sums of simpler matrices
  - $A = LU$, $A = QR$, $A = UΣV^T$, $A = \sum_i λ_i \vec{v}_i \vec{v}_i^T$, ...
The truth about mathematics

The Colbert Report on Math (February 7, 2006)
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