Scheidegger Networks—A Bonus Calculation
Complex Networks, Course 295A, Spring, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.
Outline

First return random walk

References
Random walks

- We’ve seen that Scheidegger networks have random walk boundaries \([1, 2]\)
- Determining expected shape of a ‘basin’ becomes a problem of finding the probability that a 1-d random walk returns to the origin after \(t\) time steps
- We solved this with a counting argument for the discrete random walk the preceding Complex Systems course
- For fun and the constitution, let’s work on the continuous time Wiener process version
- A classic, delightful problem
Random walks

The Wiener process (✔️)
Random walking on a sphere...

The **Wiener process** (_DEFINITION_)

[Definition Image]
Random walks

- Wiener process = Brownian motion

\[ x(t_2) - x(t_1) \sim \mathcal{N}(0, t_2 - t_1) \]

where

\[
\mathcal{N}(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}
\]

- Continuous but nowhere differentiable
First return

- **Objective:** find \( g(t) \), the probability that Wiener process first returns to the origin at time \( t \).
- Use what we know: the probability density for a return (not necessarily the first) at time \( t \) is

\[
 f(t) = \frac{1}{\sqrt{2\pi t}} e^{-0/2t} = \frac{1}{\sqrt{2\pi t}}
\]

- Observe that \( f \) and \( g \) are connected like this:

\[
 f(t) = \int_{\tau=0}^{t} f(\tau)g(t - \tau)d\tau + \delta(t)
\]

Dirac delta function

In words: Probability of returning at time \( t \) equals the integral of the probability of returning at time \( \tau \) and then not returning until exactly \( t - \tau \) time units later.
Next see that right hand side of
\[ f(t) = \int_{\tau=0}^{t} f(\tau) g(t - \tau) d\tau + \delta(t) \]
is a juicy convolution.

So we take the Laplace transform:
\[ \mathcal{L}[f(t)] = F(s) = \int_{t=0}^{\infty} f(t) e^{-st} dt \]

and obtain
\[ F(s) = F(s) G(s) + 1 \]

Rearrange:
\[ G(s) = 1 - 1/F(s) \]
First return

- We are here: $G(s) = 1 - 1/F(s)$
- Now we want to invert $G(s)$ to find $g(t)$
- Use calculation that $F(s) = (2s)^{-1/2}$

$$G(s) = 1 - (2s)^{1/2} \approx e^{-(2s)^{1/2}}$$
Groovy aspects of $g(t) \sim t^{-3/2}$:

- Variance is infinite (weird but okay...)
- Mean is also infinite (just plain crazy...)
- Distribution is normalizable so process always returns to 0.
- For river networks: $P(\ell) \sim \ell^{-\gamma}$ so $\gamma = 3/2$ for Scheidegger networks.
References I
