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Random networks

Pure, abstract random networks:
Random networks

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- Consider set of all networks with \( N \) labelled nodes and \( m \) edges.
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► Standard random network = randomly chosen network from this set.

► To be clear: each network is equally probable.

► Sometimes equiprobability is a good assumption, but it is always an assumption.

► Known as Erdös-Rényi random networks or ER graphs.
Random networks

Some features:

- Number of possible edges:

\[ 0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2} \]
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Real world: links are usually costly so real networks are almost always sparse.
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How to build standard random networks:

- Given $N$ and $m$. 
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- Given $N$ and $m$.
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1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability $p$.

2. Take $N$ nodes and add exactly $m$ links by selecting edges without replacement.
Random networks

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   - Algorithm: Randomly choose a pair of nodes $i$ and $j$, $i \neq j$, and connect if unconnected; repeat until all $m$ edges are allocated.
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   - Best for adding small numbers of links (most cases).
Random networks

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2. Two probablistic methods (we’ll see a third later on)

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   - Useful for theoretical work.
2. Take \( N \) nodes and add exactly \( m \) links by selecting edges without replacement.
   - **Algorithm:** Randomly choose a pair of nodes \( i \) and \( j \), \( i \neq j \), and connect if unconnected; repeat until all \( m \) edges are allocated.
   - Best for adding small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large \( N \).
Random networks

A few more things:

- For method 1, # links is probabilistic:

\[ \langle m \rangle = p \binom{N}{2} \]
Random networks

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- For method 1, # links is probablistic:

\[ \langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N - 1) \]
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- So the expected or average degree is

  \[ \langle k \rangle = \frac{2 \langle m \rangle}{N} \]
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\]

Which is what it should be...

- If we keep \( \langle k \rangle \) constant then \( p \propto 1/N \to 0 \) as \( N \to \infty \).
Random networks

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► Which is what it should be...

► If we keep \( \langle k \rangle \) constant then \( p \propto \frac{1}{N} \rightarrow 0 \) as \( N \rightarrow \infty \).
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Next slides:
Example realizations of random networks
Random networks: examples

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$N = 500$
Random networks: examples

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- $N = 500$
- Vary $m$, the number of edges from 100 to 1000.
Next slides:

Example realizations of random networks

- $N = 500$
- Vary $m$, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
Random networks: examples

Next slides:
Example realizations of random networks

- $N = 500$
- Vary $m$, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.
Random networks: examples

entire network:  largest component:

\[ N = 500, \text{ number of edges } m = 100 \]
\[ \text{average degree } \langle k \rangle = 0.4 \]
Random networks: examples

entire network:             largest component:

\[ N = 500, \text{ number of edges } m = 200 \]
\[ \langle k \rangle = 0.8 \]
Random networks: examples

entire network:  
largest component:

\[ N = 500, \text{ number of edges } m = 230 \]
\[ \langle k \rangle = 0.92 \]
Random networks: examples

entire network:  

largest component:  

\[N = 500, \text{ number of edges } m = 240\]
\[\text{average degree } \langle k \rangle = 0.96\]
Random networks: examples

entire network:

largest component:

\[ N = 500, \text{ number of edges } m = 250 \]
\[ \langle k \rangle = 1 \]
Random networks: examples

entire network:  
largest component:

\[ N = 500, \text{ number of edges } m = 260 \]

average degree \( \langle k \rangle = 1.04 \)
Random networks: examples

entire network:

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\[ N = 500, \text{ number of edges } m = 280 \]
\[ \langle k \rangle = 1.12 \]
Random networks: examples

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\( N = 500 \), number of edges \( m = 300 \)
average degree \( \langle k \rangle = 1.2 \)
Random networks: examples

entire network: 

largest component: 

\[ N = 500, \text{ number of edges } m = 500 \]
\[ \langle k \rangle = 2 \]
Random networks: examples

entire network:  

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\[ N = 500, \text{ number of edges } m = 1000 \]
\[ \langle k \rangle = 4 \]
Random networks: examples for $N=500$

$m = 100$
$\langle k \rangle = 0.4$

$m = 200$
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$m = 230$
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$m = 240$
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Clustering:

▶ For method 1, what is the clustering coefficient for a finite network?

Consider triangle/triple clustering coefficient (Newman [1]):

\[ C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}} \]

Recall:

\[ C_2 = \text{probability that two nodes are connected given they have a friend in common.} \]

For standard random networks, we have simply that

\[ C_2 = p. \]
Random networks

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Clustering:

- So for large random networks ($N \to \infty$), clustering drops to zero.
Random networks

Clustering:

- So for large random networks ($N \to \infty$), clustering drops to zero.
- Key structural feature of random networks is that they locally look like branching networks (no loops).
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- Recall $p_k$ = probability that a randomly selected node has degree $k$. 
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- Now consider one node: there are ‘$N$ choose $k$’ ways the node can be connected to $k$ of the other $N - 1$ nodes.
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- Each connection occurs with probability $p$, each non-connection with probability $(1 - p)$. 
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Degree distribution:

- Recall $p_k$ = probability that a randomly selected node has degree $k$.
- Consider method 1 for constructing random networks: each possible link is realized with probability $p$.
- Now consider one node: there are ‘$N$ choose $k$’ ways the node can be connected to $k$ of the other $N - 1$ nodes.
- Each connection occurs with probability $p$, each non-connection with probability $(1 - p)$.
- Therefore have a binomial distribution:

$$P(k; p, N) = \binom{N - 1}{k} p^k (1 - p)^{N-1-k}.$$
Random networks

Limiting form of $P(k; p, N)$:

Our degree distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-k-1}.$$ 

What happens as $N \to \infty$?

We must end up with the normal distribution right?

If $p$ is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \approx pN \to \infty$.

But we want to keep $\langle k \rangle$ fixed...

So examine limit of $P(k; p, N)$ when $p \to 0$ and $N \to \infty$ with $\langle k \rangle = p(N-1) = \text{constant}$. 

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Limiting form of $P(k; p, N)$:

- Substitute $p = \frac{\langle k \rangle}{N-1}$ into $P(k; p, N)$ and hold $k$ fixed:

$$P(k; p, N) = \binom{N-1}{k} \left( \frac{\langle k \rangle}{N-1} \right)^k \left( 1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k}$$
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$$= \frac{N^k(1 - \frac{1}{N}) \cdots (1 - \frac{k}{N})}{k! N^k} \frac{\langle k \rangle^k}{(1 - \frac{1}{N})^k} \left( 1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k}$$
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\[
= \frac{(N - 1)!}{k!(N - 1 - k)!} \frac{\langle k \rangle^k}{(N - 1)^k} \left( 1 - \frac{\langle k \rangle}{N - 1} \right)^{N-1-k}
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\[
= \frac{(N - 1)(N - 2) \ldots (N - k)}{k!} \frac{\langle k \rangle^k}{(N - 1)^k} \left( 1 - \frac{\langle k \rangle}{N - 1} \right)^{N-1-k}
\]

\[
= \frac{N^k(1 - \frac{1}{N}) \cdots (1 - \frac{k}{N})}{k!N^k} \frac{\langle k \rangle^k}{(1 - \frac{1}{N})^k} \left( 1 - \frac{\langle k \rangle}{N - 1} \right)^{N-1-k}
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Limiting form of $P(k; p, N)$:

- We are now here:

$$P(k; p, N) \sim \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N - 1}\right)^{N-1-k}$$
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- Now use the excellent result:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$
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(Use l’Hôpital’s rule to prove.)

- Identifying $n = N - 1$ and $x = -\langle k \rangle$:

$$P(k; \langle k \rangle) \sim \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \left(1 - \frac{\langle k \rangle}{N - 1}\right)^{-k}$$
Limiting form of $P(k; p, N)$:

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- This is a Poisson distribution (⊞) with mean $\langle k \rangle$. 
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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:
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- Apart from degree distribution, wiring is random.
Random networks: examples for $N=1000$

- $\gamma = 2.1$, $\langle k \rangle = 3.448$
- $\gamma = 2.19$, $\langle k \rangle = 2.986$
- $\gamma = 2.28$, $\langle k \rangle = 2.306$
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Random networks: largest components

\begin{align*}
\gamma &= 2.1 \\
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- The **variance** of degree distributions for random networks turns out to be **very important**.
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- Use calculation similar to one for finding $\langle k \rangle$ to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$
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The edge-degree distribution:

The degree distribution $P_k$ is fundamental for our description of many complex networks.

Again: $P_k$ is the degree of a randomly chosen node.

A second very important distribution arises from choosing randomly on edges rather than on nodes. Define $Q_k$ to be the probability the node at a random end of a randomly chosen edge has degree $k$.

Now choosing nodes based on their degree (i.e., size): $Q_k \propto k P_k$.

Normalized form: $Q_k = \sum_{k'=0}^{\infty} k P_{k'} = \langle k \rangle k P_k$. 

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- Equivalent to friend having degree $k + 1$.
- **Natural question**: what’s the expected number of other friends that one friend has?
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- Given $R_k$ is the probability that a friend has $k$ other friends, then the average number of friends’ other friends is

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- Again, neatness of results is a special property of the Poisson distribution.
The edge-degree distribution:

- Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle} \left( \langle k^2 \rangle - \langle k \rangle \right)$, is true for all random networks, independent of degree distribution.
- For standard random networks, recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle} \left( \langle k^2 \rangle + \langle k \rangle - \langle k \rangle \right) = \langle k \rangle$$

- Again, neatness of results is a special property of the Poisson distribution.
- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...
Two reasons why this matters

Reason #1:

\[
\langle k^2 \rangle = \langle k \rangle \times \langle k \rangle \quad R = \langle k \rangle^{1/2} \langle k \rangle \left( \langle k^2 \rangle - \langle k \rangle \right) = \langle k^2 \rangle - \langle k \rangle.
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Key: Average depends on the 1st and 2nd moments of \( P_k \) and not just the 1st moment.

Three peculiarities:
1. We might guess \( \langle k^2 \rangle = \langle k \rangle \left( \langle k \rangle - 1 \right) \) but it's actually \( \langle k(k-1) \rangle \).
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3. Your friends are different to you...
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Reason #1:

- Average # friends of friends per node is

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- So only if everyone has the same degree (variance $\sigma^2 = 0$) can a node be the same as its friends.

Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.
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(Big) Reason #2:

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- Note: Component = Cluster
Outline

Basics
  - Definitions
  - How to build
  - Some visual examples

Structure
  - Clustering
  - Degree distributions
  - Configuration model
    - Largest component

Generating Functions
  - Definitions
  - Basic Properties
  - Giant Component Condition
  - Component sizes
  - Useful results
  - Size of the Giant Component
  - Average Component Size

References
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Giant component condition (or percolation condition):

$$\langle k \rangle_R = \langle k^2 \rangle - \langle k \rangle \langle k \rangle > 1$$

Again, see that the second moment is an essential part of the story.

Equivalent statement:

$$\langle k^2 \rangle > 2 \langle k \rangle$$
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- e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$ then

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And how big is the largest component?

- Define $S_1$ as the size of the largest component.
Giant component

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- Define $S_1$ as the size of the largest component.
- Consider an infinite ER random network with average degree $\langle k \rangle$. 

Let's find $S_1$ with a back-of-the-envelope argument.

Define $\delta$ as the probability that a randomly chosen node does not belong to the largest component.

Simple connection: $\delta = 1 - S_1$.

Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.

So $\delta = \infty \sum k = 0 P_k \delta_k$.

Substitute in Poisson distribution...
Giant component

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Carrying on:

\[ \delta = \sum_{k=0}^{\infty} P_k \delta^k \]
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Now substitute in \( \delta = 1 - S_1 \) and rearrange to obtain:

\[ S_1 = 1 - e^{-\langle k \rangle} S_1. \]
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We can figure out some limits and details for \( S_1 = 1 - e^{-\langle k \rangle S_1} \).
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- First, we can write \( \langle k \rangle \) in terms of \( S_1 \):
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References
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$S_1$ vs. $\langle k \rangle$
Giant component

Turns out we were lucky...

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- We can do this but we need to enhance our toolkit with Generatingfunctionology... [3]
Generating functions

- **Idea:** Given a sequence $a_0, a_1, a_2, \ldots$, associate each element with a distinct function or other mathematical object.
Generating functions

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- The **generating function (g.f.)** for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$
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► Related to Fourier, Laplace, Mellin, \ldots
Example

- Take a degree distribution with exponential decay:

\[ P_k = ce^{-\lambda k} \]

where \( c = 1 - e^{-\lambda} \).
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Outline

Basics
Definitions
How to build
Some visual examples

Structure
Clustering
Degree distributions
Configuration model
Largest component

Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
Average Component Size

References
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- Average degree:

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\[ \langle k \rangle = \sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{\infty} kP_k x^{k-1} \bigg|_{x=1} \]

In general, many calculations become simple, if a little abstract.

For our exponential example:

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So:

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Useful pieces for probability distributions:

Normalization:
\[ F(1) = 1 \]

First moment:
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- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$
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- Now find how \( F_R \) is related to \( F_P \)…
Edge-degree distribution

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\[ F_R(x) = \sum_{k=0}^{\infty} R_k x^k \]
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\[ F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k + 1) P_{k+1}}{\langle k \rangle} x^k. \]
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Finally, since \( \langle k \rangle = F'_P(1) \),

\[ F_R(x) = \frac{F'_P(x)}{F'_P(1)} \]
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- Recall giant component condition is
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  \]
- Setting \( x = 1 \), our condition becomes
  \[
  \frac{F''_P(1)}{F'_P(1)} > 1.
  \]
Size distributions

To figure out the size of the largest component \( S_1 \), we need more resolution on component sizes.
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Definitions:

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Local-global connection:

$$P_k, R_k \iff \pi_n, \rho_n$$

neighbors $\iff$ components
Size distributions

G.f.’s for component size distributions:

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\[ \Phi_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \]

\[ \Phi_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n \]

The largest component:

Subtle key: \( \Phi_{\pi}(1) \) is the probability that a node belongs to a finite component.

Therefore:

\[ S_1 = 1 - \Phi_{\pi}(1) \]

Our mission, which we accept:

Find the four generating functions \( \Phi_P, \Phi_R, \Phi_{\pi}, \) and \( \Phi_{\rho} \).
Size distributions

G.f.’s for component size distributions:

\[ F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \quad \text{and} \quad F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n \]
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\[ F_P, F_R, F_\pi, \text{ and } F_\rho. \]
Useful results we’ll need for g.f.’s

Sneaky Result 1:

Consider two random variables $U$ and $V$ whose values may be 0, 1, 2, ... Write probability distributions as $U_k$ and $V_k$ and g.f.’s as $F_U$ and $F_V$. SR1: If a third random variable is defined as $W = \sum_{i=1}^{\infty} U_i(V_i)$ then $F_W(x) = F_V(F_U(x))$. 
Useful results we’ll need for g.f.’s

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▶ \textbf{SR1} If a third random variable is defined as $W = \sum_{i=1}^{\infty} U_i(V_i)$ with each $U_i(V_i)$ having the same distribution as $U(V)$, then $F_W(x) = F_V(F_U(x))$. 

Useful results we’ll need for g.f.’s

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- Consider two random variables $U$ and $V$ whose values may be $0, 1, 2, \ldots$
- Write probability distributions as $U_k$ and $V_k$ and g.f.’s as $F_U$ and $F_V$.
- SR1: If a third random variable is defined as

\[ W = \sum_{i=1}^{V} U^{(i)} \text{ with each } U^{(i)} \overset{d}{=} U \]
Useful results we’ll need for g.f.’s

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► Consider two random variables $U$ and $V$ whose values may be $0, 1, 2, \ldots$
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Proof of SN1:

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$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k$$
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Proof of SN1:

With some concentration, observe:

\[
F_W(x) = \sum_{j=0}^{\infty} V_j \sum_{k=0}^{\infty} \sum_{{\{i_1, i_2, \ldots, i_k\}} \mid i_1 + i_2 + \ldots + i_k = j} U_{i_1} x^{i_1} U_{i_2} x^{i_2} \cdots U_{i_j} x^{i_j} \\
x^k \text{ piece of } \left( \sum_{i'=0}^{\infty} U_{i'} x^{i'} \right)^j
\]
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\]

\[
\underbrace{x^k \text{ piece of } \left( \sum_{i'=0}^{\infty} U_{i'} x^{i'} \right)^j}_{\text{expression}} = (F_U(x))^j
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\]

\[
= F_V (F_U(x)) \checkmark
\]
Useful results we’ll need for g.f.’s

Sneaky Result 2:

Reason:

\[ F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_k x^{k-1} = x \sum_{j=0}^{\infty} U_j x^j = x F_U(x). \]
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- $\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k$
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Generalization of SN2:

\(F_V(x) = x^i F_U(x)\)

\(F_V(x) = x - i (F_U(x) - U_0 - U_1 x - \ldots - U_{i-1} x^{i-1})\)

\[\sum_{k=i}^{\infty} k U_k x^k\]
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   \[ = x^{-i} \sum_{k=i}^{\infty} U_k x^k \]
Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$. 

- Probability that a random node belongs to a finite component of size $n$: 

$$\pi_n = \sum_{k=0}^{\infty} P_k \times \Pr(\text{sum of sizes of subcomponents at end of } k \text{ random links} = n - 1)$$

- Therefore: 

$$F_\pi(x) = x \cdot F_\rho(F_\pi(x)) \cdot SN_2$$

- Extra factor of $x$ accounts for random node itself.
Goal: figure out forms of the component generating functions, $F_\pi$ and $F_\rho$.

$\pi_n = \text{probability that a random node belongs to a finite component of size } n$
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\[
F_\pi(x) = F_P(F_\rho(x)) \quad \text{SN1}
\]
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\]

- Therefore:

\[
F_\pi(x) = x \frac{F_P(F_\rho(x))}{SN2} \frac{SN1}{SN}
\]
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\]

Therefore:

\[
F_\pi(x) = x \underbrace{F_\rho(F_\rho(x))}_{SN1} \underbrace{SN2}_{x}
\]

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\( \rho_n \) = probability that a random link leads to a finite subcomponent of size \( n \).
Connecting generating functions

- $\rho_n = \text{probability that a random link leads to a finite subcomponent of size } n$.
- Invoke one step of recursion: $\rho_n = \text{probability that a random node arrived along a random edge is part of a finite subcomponent of size } n$. 

\[ F(\rho(x)) = \frac{1}{x} \sum_{k=0}^{\infty} R_k \times \text{Pr}(\text{sum of sizes of subcomponents at end of } k \text{ random links } = n-1) \]

Therefore:

\[ F(\rho(x)) = x \cdot F(R(F(\rho(x)))) \]

Again, extra factor of $x$ accounts for random node itself.
Connecting generating functions

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$$\sum_{k=0}^{\infty} R_k \times \Pr \left( \text{sum of sizes of subcomponents at end of } k \text{ random links } = n - 1 \right)$$

Therefore:

$$F_{\rho}(x) = x \cdot \frac{F_R(F_{\rho}(x))}{SN2}$$

$$\frac{1}{SN1}$$
Connecting generating functions

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\]

Therefore:

\[
F_\rho(x) = x \underbrace{F_R(F_\rho(x))}_{\text{SN2}} \underbrace{F_R}_{\text{SN1}}
\]

- Again, extra factor of $x$ accounts for random node itself.
We now have two functional equations connecting our generating functions:

\[ F_\pi(x) = x F_P (F_\rho(x)) \quad \text{and} \quad F_\rho(x) = x F_R (F_\rho(x)) \]
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- We first untangle the second equation to find \( F_\rho \).
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We can do this because it only involves \( F_\rho \) and \( F_R \).
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- The first equation then immediately gives us \( F_{\pi} \) in terms of \( F_\rho \) and \( F_R \).
Component sizes

- Remembering vaguely what we are doing:
Component sizes

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  Finding $F_P$ to obtain the size of the largest component $S_1 = 1 - F_\pi(1)$. 

References
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  \[
  F_\pi(1) = F_P(F_\rho(1)) \quad \text{and} \quad F_\rho(1) = F_R(F_\rho(1))
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- Solve second equation numerically for $F_{\rho}(1)$. 
Component sizes

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  - Finding $F_P$ to obtain the size of the largest component $S_1 = 1 - F_\pi(1)$.
  - Set $x = 1$ in our two equations:
    
    $$F_\pi(1) = F_P(F_\rho(1)) \quad \text{and} \quad F_\rho(1) = F_R(F_\rho(1))$$

- Solve second equation numerically for $F_\rho(1)$.
- Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$. 
Component sizes

**Example:** Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$
Component sizes

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- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

  $\therefore F_R(x) = F'_P(x)/F'_P(1)$
Component sizes

**Example: Standard random graphs.**

- We can show \( F_P(x) = e^{-\langle k \rangle (1-x)} \)

\[
\therefore F_R(x) = \frac{F'_P(x)}{F'_P(1)} = \frac{e^{-\langle k \rangle (1-x)}}{e^{-\langle k \rangle (1-x')}} \bigg|_{x'=1}
\]

aha! RHS's of our two equations are the same. So \( F_\pi(x) = \frac{x F_R(F_\rho(x))}{F_R(F_\pi(x))} \)

Why our dirty (but wrong) trick worked earlier...
Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

\[
\therefore F_R(x) = \frac{F_P'(x)}{F_P'(1)} = \frac{e^{-\langle k \rangle (1-x)}}{e^{-\langle k \rangle (1-x')}} \bigg|_{x'=1} = e^{-\langle k \rangle (1-x)}
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\[
\therefore F_R(x) = \frac{F'_P(x)}{F'_P(1)} = \frac{e^{-\langle k \rangle (1-x)}}{e^{-\langle k \rangle (1-x')}} \bigg|_{x'=1} = e^{-\langle k \rangle (1-x)} = F_P(x) \quad \text{...aha!}
\]
Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

\[
\begin{align*}
\therefore F_R(x) &= F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1} \\
&= e^{-\langle k \rangle (1-x)} = F_P(x) \quad \text{...aha!}
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- RHS’s of our two equations are the same.
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Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

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- Again, have to resort to numerics at this point.
Outline

Basics
  Definitions
  How to build
  Some visual examples
Structure
  Clustering
  Degree distributions
  Configuration model
  Largest component
Generating Functions
  Definitions
  Basic Properties
  Giant Component Condition
  Component sizes
  Useful results
  Size of the Giant Component
  Average Component Size
References
Average component size

Next: find average size of finite components $\langle n \rangle$. 
Average component size

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- Using standard G.F. result: \( \langle n \rangle = F'_\pi(1) \).
- Try to avoid finding \( F_\pi(x) \)...
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Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.

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- Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$. 
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Example: Standard random graphs.
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End result:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$
Average component size

- Our result for standard random networks:

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- **Reason:** we have a power law distribution of component sizes at \( \langle k \rangle = 1 \).
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- Typical critical point behavior....
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- Limits of $\langle k \rangle = 0$ and $\infty$ make sense for

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- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$. 

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All nodes are isolated.

No nodes are outside of the giant component.
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