Outline

Introduction

River Networks
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  Allometry
  Laws
  Stream Ordering
  Horton’s Laws
  Tokunaga’s Law
  Horton ⇔ Tokunaga
  Reducing Horton
  Scaling relations
  Fluctuations
  Models

References
Introduction

Branching networks are useful things:

- Fundamental to material *supply and collection*
- **Supply:** From one source to many sinks in 2- or 3-d.
- **Collection:** From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

Examples:

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

References
Branching networks are everywhere...

http://hydrosheds.cr.usgs.gov/
Branching networks are everywhere...

Geomorphological networks

Definitions

- **Drainage basin** for a point $p$ is the complete region of land from which overland flow drains through $p$.
- Definition most sensible for a point in a stream.
- **Recursive structure**: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
- Okay for large-scale networks...
Basic basin quantities: $a$, $l$, $L_{\parallel}$, $L_{\perp}$:

- $a$ = drainage basin area
- $l$ = length of longest (main) stream (which may be fractal)
- $L = L_{\parallel} = \text{longitudinal length of basin}$
- $L = L_{\perp} = \text{width of basin}$
Isometry: dimensions scale linearly with each other.

Allometry: dimensions scale non-linearly.
Basin allometry

Allometric relationships:

\[ l \propto a^h \]

\[ l \propto L^d \]

Combine above:

\[ a \propto L^{d/h} \equiv L^D \]
‘Laws’

- Hack’s law (1957) \([6]\):
  \[
  \ell \propto a^h
  \]
  reportedly \(0.5 < h < 0.7\)

- Scaling of main stream length with basin size:
  \[
  \ell \propto L^d
  \]
  reportedly \(1.0 < d < 1.1\)

- Basin allometry:
  \[
  L^\parallel \propto a^{h/d} \equiv a^{1/D}
  \]
  \(D < 2 \rightarrow \) basins elongate.
There are a few more ‘laws’: [2]

<table>
<thead>
<tr>
<th>Relation:</th>
<th>Name or description:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_k = T_1(R_T)^k$</td>
<td>Tokunaga’s law</td>
</tr>
<tr>
<td>$\ell \sim L^d$</td>
<td>self-affinity of single channels</td>
</tr>
<tr>
<td>$n_\omega/n_{\omega+1} = R_n$</td>
<td>Horton’s law of stream numbers</td>
</tr>
<tr>
<td>$\bar{\ell}<em>{\omega+1}/\bar{\ell}</em>\omega = R_\ell$</td>
<td>Horton’s law of main stream lengths</td>
</tr>
<tr>
<td>$\bar{a}<em>{\omega+1}/\bar{a}</em>\omega = R_a$</td>
<td>Horton’s law of basin areas</td>
</tr>
<tr>
<td>$\bar{s}<em>{\omega+1}/\bar{s}</em>\omega = R_s$</td>
<td>Horton’s law of stream segment lengths</td>
</tr>
<tr>
<td>$L_{\perp} \sim L^H$</td>
<td>scaling of basin widths</td>
</tr>
<tr>
<td>$P(a) \sim a^{-\tau}$</td>
<td>probability of basin areas</td>
</tr>
<tr>
<td>$P(\ell) \sim \ell^{-\gamma}$</td>
<td>probability of stream lengths</td>
</tr>
<tr>
<td>$\ell \sim a^h$</td>
<td>Hack’s law</td>
</tr>
<tr>
<td>$a \sim L^D$</td>
<td>scaling of basin areas</td>
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<tr>
<td>$\Lambda \sim a^\beta$</td>
<td>Langbein’s law</td>
</tr>
<tr>
<td>$\lambda \sim L^\phi$</td>
<td>variation of Langbein’s law</td>
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## Reported parameter values: [2]

<table>
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<tr>
<th>Parameter</th>
<th>Real networks:</th>
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<tr>
<td>$R_n$</td>
<td>3.0–5.0</td>
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<tr>
<td>$R_a$</td>
<td>3.0–6.0</td>
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<tr>
<td>$R_\ell = R_T$</td>
<td>1.5–3.0</td>
</tr>
<tr>
<td>$T_1$</td>
<td>1.0–1.5</td>
</tr>
<tr>
<td>$d$</td>
<td>1.1 ± 0.01</td>
</tr>
<tr>
<td>$D$</td>
<td>1.8 ± 0.1</td>
</tr>
<tr>
<td>$h$</td>
<td>0.50–0.70</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1.43 ± 0.05</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.8 ± 0.1</td>
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<tr>
<td>$H$</td>
<td>0.75–0.80</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.50–0.70</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.05 ± 0.05</td>
</tr>
</tbody>
</table>
Kind of a mess...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out...
Stream Ordering:

Method for describing network architecture:

- Introduced by Horton (1945) \(^7\)
- Modified by Strahler (1957) \(^{16}\)
- Term: Horton-Strahler Stream Ordering \(^{11}\)
- Can be seen as \textit{iterative trimming} of a network.
Stream Ordering:

Some definitions:

- A **channel head** is a point in landscape where flow becomes focused enough to form a stream.
- A **source stream** is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.
Stream Ordering:

1. Label all source streams as order $\omega = 1$ and remove.
2. Label all new source streams as order $\omega = 2$ and remove.
3. Repeat until one stream is left (order $= \Omega$)
4. Basin is said to be of the order of the last stream removed.
5. Example above is a basin of order $\Omega = 3$. 
Stream Ordering—A large example:
Stream Ordering:

Another way to define ordering:

- As before, label all **source streams** as order $\omega = 1$.
- Follow all labelled streams downstream.
- Whenever two streams of the same order ($\omega$) meet, the resulting stream has order incremented by 1 ($\omega + 1$).

- If streams of different orders $\omega_1$ and $\omega_2$ meet, then the resultant stream has order equal to the largest of the two.

- Simple rule:
  
  $$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

  where $\delta$ is the Kronecker delta.
Stream Ordering:

One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ... but relationships based on ordering appear to be robust to resolution changes.
Stream Ordering:

Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand network architecture
Stream Ordering:

Resultant definitions:

- A basin of order $\Omega$ has $n_\omega$ streams (or sub-basins) of order $\omega$.
  - $n_\omega > n_{\omega+1}$
- An order $\omega$ basin has area $a_\omega$.
- An order $\omega$ basin has a main stream length $l_\omega$.
- An order $\omega$ basin has a stream segment length $s_\omega$
  1. an order $\omega$ stream segment is only that part of the stream which is actually of order $\omega$
  2. an order $\omega$ stream segment runs from the basin outlet up to the junction of two order $\omega - 1$ streams
Horton’s laws

Self-similarity of river networks

- First quantified by Horton (1945) \(^7\), expanded by Schumm (1956) \(^{14}\)

Three laws:

- Horton’s law of stream numbers:
  \[
  \frac{n_\omega}{n_{\omega+1}} = R_n > 1
  \]

- Horton’s law of stream lengths:
  \[
  \frac{\bar{\ell}_{\omega+1}}{\bar{\ell}_\omega} = R_\ell > 1
  \]

- Horton’s law of basin areas:
  \[
  \frac{\bar{a}_{\omega+1}}{\bar{a}_\omega} = R_a > 1
  \]
Horton’s laws

Horton’s Ratios:

- So... Horton’s laws are defined by three ratios:

\[ R_n, \ R_\ell, \ \text{and} \ R_a. \]

- Horton’s laws describe exponential decay or growth:

\[
\begin{align*}
  n_\omega &= n_{\omega-1} / R_n \\
  &= n_{\omega-2} / R_n^2 \\
  &\vdots \\
  &= n_1 / R_n^{\omega-1} \\
  &= n_1 e^{- (\omega-1) \ln R_n}
\end{align*}
\]
Horton’s laws

Similar story for area and length:

\[ \bar{a}_\omega = \bar{a}_1 e^{(\omega - 1) \ln R_a} \]

\[ \bar{\ell}_\omega = \bar{\ell}_1 e^{(\omega - 1) \ln R_\ell} \]

As stream order increases, number drops and area and length increase.
Horton’s laws

A few more things:

- Horton’s laws are laws of averages.
- Averaging for number is **across** basins.
- Averaging for stream lengths and areas is **within** basins.
- Horton’s ratios go a long way to defining a branching network...
- But we need one other piece of information...
Horton’s laws

A bonus law:

- Horton’s law of stream segment lengths:
  \[
  \frac{\bar{s}_{\omega+1}}{\bar{s}_\omega} = R_s > 1
  \]

- Can show that \( R_s = R_\ell \).
Horton’s laws in the real world:

The Mississippi

The Nile

The Amazon

References
Horton’s laws-at-large

Blood networks:
- Horton’s laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...
- Vessel diameters obey an analogous Horton’s law.
Horton’s laws

Observations:

- Horton’s ratios vary:
  - $R_n$: 3.0–5.0
  - $R_a$: 3.0–6.0
  - $R_\ell$: 1.5–3.0

- No accepted explanation for these values.
- Horton’s laws tell us how quantities vary from level to level ...
- ... but they don’t explain how networks are structured.
Tokunaga’s law

Delving deeper into network architecture:

- Tokunaga (1968) identified a clearer picture of network structure \([21, 22, 23]\).
- As per Horton-Strahler, use stream ordering.
- **Focus**: describe how streams of different orders connect to each other.
- Tokunaga’s law is also a law of averages.
Network Architecture

Definition:

- $T_{\mu, \nu} =$ the average number of side streams of order $\nu$ that enter as tributaries to streams of order $\mu$
- $\mu, \nu = 1, 2, 3, \ldots$
- $\mu \geq \nu + 1$
- Recall each stream segment of order $\mu$ is ‘generated’ by two streams of order $\mu - 1$
- These generating streams are not considered side streams.
Network Architecture

Tokunaga’s law

- Property 1: Scale independence—depends only on difference between orders:
  \[ T_{\mu,\nu} = T_{\mu-\nu} \]

- Property 2: Number of side streams grows exponentially with difference in orders:
  \[ T_{\mu,\nu} = T_1 (R_T)^{\mu-\nu-1} \]

- We usually write Tokunaga’s law as:
  \[ T_k = T_1 (R_T)^{k-1} \quad \text{where } R_T \approx 2 \]
Tokunaga’s law—an example:

\[
T_1 \approx 2 \\
R_T \approx 4
\]
The Mississippi

A Tokunaga graph:

\[ \log_{10} \langle T_{\mu, \nu} \rangle \]

\[ \nu = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \]
Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- Horton’s laws appear to contain less detailed information than Tokunaga’s law.
- Oddly, Horton’s law has three parameters and Tokunaga has two parameters.
- $R_n$, $R_\ell$, and $R_s$ versus $T_1$ and $R_T$.
- To make a connection, clearest approach is to start with Tokunaga’s law...
- Known result: Tokunaga $\rightarrow$ Horton [21, 22, 23, 10, 2]
Let us make them happy

We need one more ingredient:

Space-fillingness

► A network is **space-filling** if the average distance between adjacent streams is roughly constant.
► Reasonable for river and cardiovascular networks
► For river networks:
  **Drainage density** $\rho_{dd} =$ inverse of typical distance between channels in a landscape.
► In terms of basin characteristics:

$$\rho_{dd} \approx \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \sum_{\omega=1}^{\Omega} \frac{n_\omega s_\omega}{a_\Omega}$$
More with the happy-making thing

Start with Tokunaga’s law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton’s stream number law:
  $$n_\omega / n_{\omega+1} = R_n.$$  
- Estimate $n_\omega$, the number of streams of order $\omega$ in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order $\omega$ terminates by either:
  
1. Running into another stream of order $\omega$ and generating a stream of order $\omega + 1$...
   - $2n_{\omega+1}$ streams of order $\omega$ do this

2. Running into and being absorbed by a stream of higher order $\omega' > \omega$...
   - $n'_{\omega'} T_{\omega'-\omega}$ streams of order $\omega$ do this
More with the happy-making thing

Putting things together:

\[ n_\omega = 2n_{\omega+1} + \sum_{\omega' = \omega+1}^{\omega} T_{\omega' - \omega} n_{\omega'} \]

\[ n_\omega = \sum_{\omega' = \omega+1}^{\omega} T_{\omega' - \omega} n_{\omega'} \]

- Substitute in \( T_{\omega' - \omega} = T_1(R_T)^{\omega' - \omega - 1} \):

\[ n_\omega = 2n_{\omega+1} + \sum_{\omega' = \omega+1}^{\omega} T_1(R_T)^{\omega' - \omega - 1} n_{\omega'} \]

- Shift index to \( k = \omega' - \omega \):

\[ n_\omega = 2n_{\omega+1} + \sum_{k=1}^{\Omega - \omega} T_1(R_T)^{k-1} n_{\omega+k} \]
More with the happy-making thing

Create Horton ratios:

- Divide through by $n_{\omega+1}$:

\[
\frac{n_{\omega}}{n_{\omega+1}} = \frac{2n_{\omega+1}}{n_{\omega+1}} + \sum_{k=1}^{\Omega-\omega} T_1(R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}
\]

- Left hand side looks good but we have $n_{\omega+k}/n_{\omega+1}$'s hanging around on the right.

- Recall, we want to show $R_n = n_{\omega}/n_{\omega+1}$ is a constant, independent of $\omega$. ...
More with the happy-making thing

Finding Horton ratios:

- Letting $\Omega \rightarrow \infty$, we have

\[
\frac{n_\omega}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}
\]  

(1)

- The ratio $n_{\omega+k} / n_{\omega+1}$ can only be a function of $k$ due to self-similarity (which is implicit in Tokunaga’s law).

- The ratio $n_\omega / n_{\omega+1}$ is independent of $\omega$ and depends only on $T_1$ and $R_T$.

- Can now call $n_\omega / n_{\omega+1} = R_n$.

- Immediately have $n_{\omega+k} / n_{\omega+1} = R_n^{-(k-1)}$.

- Plug into Eq. (1)...
More with the happy-making thing

Finding Horton ratios:

- Now have:

\[
R_n = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} R_n^{-(k-1)} \\
= 2 + T_1 \sum_{k=1}^{\infty} \left( \frac{R_T}{R_n} \right)^{k-1} \\
= 2 + T_1 \frac{1}{1 - R_T/R_n}
\]

- Rearrange to find:

\[
(R_n - 2)(1 - R_T/R_n) = T_1
\]
More with the happy-making thing

Finding $R_n$ in terms of $T_1$ and $R_T$:

- **We are here:** $(R_n - 2)(1 - R_T/R_n) = T_1$
- **\times R_n to find quadratic in $R_n$:**

\[
(R_n - 2)(R_n - R_T) = T_1 R_n
\]

- **\[
R_n^2 - (2 + R_T + T_1)R_n + 2R_T = 0
\]

- **Solution:**

\[
R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}
\]
Finding other Horton ratios

Connect Tokunaga to $R_s$

- Now use uniform drainage density $\rho_{dd}$.
- Assume side streams are roughly separated by distance $1/\rho_{dd}$.
- For an order $\omega$ stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left(1 + \sum_{k=1}^{\omega-1} T_k\right)$$

- Substitute in Tokunaga’s law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{dd}^{-1} \left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1}\right) \propto R_T^\omega$$
Horton and Tokunaga are happy

Altogether then:

$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_S = R_T$

- Recall $R_\ell = R_S$ so

$R_\ell = R_T$

- And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$
Horton and Tokunaga are happy

Some observations:

- \( R_n \) and \( R_\ell \) depend on \( T_1 \) and \( R_T \).
- Seems that \( R_a \) must as well...
- Suggests Horton’s laws must contain some redundancy
- We’ll in fact see that \( R_a = R_n \).
- Also: Both Tokunaga’s law and Horton’s laws can be generalized to relationships between statistical distributions. \(^{[3, 4]}\)
Horton and Tokunaga are happy

The other way round

- Note: We can invert the expressions for $R_n$ and $R_\ell$ to find Tokunaga’s parameters in terms of Horton’s parameters.

$$R_T = R_\ell,$$

$$T_1 = R_n - R_\ell - 2 + 2R_\ell / R_n.$$  

- Suggests we should be able to argue that Horton’s laws imply Tokunaga’s laws (if drainage density is uniform)...
Horton and Tokunaga are friends

From Horton to Tokunaga \(^2\)

- Assume Horton’s laws hold for number and length
- Start with an order \(\omega\) stream
- Scale up by a factor of \(R_\ell\), orders increment
- Maintain drainage density by adding new order 1 streams

Assume Horton’s laws hold for number and length

Start with an order \(\omega\) stream

Scale up by a factor of \(R_\ell\), orders increment

Maintain drainage density by adding new order 1 streams
Horton and Tokunaga are friends

... and in detail:

- Must retain same drainage density.
- Add an extra \((R_\ell - 1)\) first order streams for each original tributary.
- Since number of first order streams is now given by \(T_{k+1}\) we have:

\[
T_{k+1} = (R_\ell - 1) \left( \sum_{i=1}^{k} T_i + 1 \right).
\]

- For large \(\omega\), Tokunaga’s law is the solution—let’s check...
Horton and Tokunaga are friends

Just checking:

 strawberries

 - Substitute Tokunaga’s law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

$$T_{k+1} = (R_\ell - 1) \left( \sum_{i=1}^{k} T_i + 1 \right)$$

 - Then:

$$T_{k+1} = (R_\ell - 1) \left( \sum_{i=1}^{k} T_1 R_\ell^{i-1} + 1 \right)$$

$$= (R_\ell - 1) T_1 \left( \frac{R_\ell^k - 1}{R_\ell - 1} + 1 \right)$$

$$\simeq (R_\ell - 1) T_1 \frac{R_\ell^k}{R_\ell - 1} = T_1 R_\ell^k \quad \ldots \text{ yep.}$$
Horton’s laws of area and number:

In right plots, stream number graph has been flipped vertically.

Highly suggestive that $R_n \equiv R_a$...
Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.
### Mississippi:

<table>
<thead>
<tr>
<th>ω range</th>
<th>$R_n$</th>
<th>$R_a$</th>
<th>$R_\ell$</th>
<th>$R_s$</th>
<th>$R_a/R_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2, 3]</td>
<td>5.27</td>
<td>5.26</td>
<td>2.48</td>
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<td>1.00</td>
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<td>[2, 5]</td>
<td>4.86</td>
<td>4.96</td>
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<td>[3, 4]</td>
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<tr>
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<td>0.04</td>
<td>0.07</td>
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<tr>
<td>$\sigma/\mu$</td>
<td>0.045</td>
<td>0.027</td>
<td>0.015</td>
<td>0.031</td>
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### Amazon:

<table>
<thead>
<tr>
<th>ω range</th>
<th>$R_n$</th>
<th>$R_a$</th>
<th>$R_\ell$</th>
<th>$R_s$</th>
<th>$R_a/R_n$</th>
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<td>[2, 3]</td>
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<td>1.01</td>
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<tr>
<td>[2, 7]</td>
<td>4.42</td>
<td>4.53</td>
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<tr>
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</tr>
<tr>
<td>[3, 7]</td>
<td>4.35</td>
<td>4.49</td>
<td>2.20</td>
<td>2.10</td>
<td>1.03</td>
</tr>
<tr>
<td>[4, 6]</td>
<td>4.38</td>
<td>4.54</td>
<td>2.22</td>
<td>2.18</td>
<td>1.03</td>
</tr>
<tr>
<td>[5, 6]</td>
<td>4.38</td>
<td>4.62</td>
<td>2.22</td>
<td>2.21</td>
<td>1.06</td>
</tr>
<tr>
<td>[6, 7]</td>
<td>4.08</td>
<td>4.27</td>
<td>2.05</td>
<td>1.83</td>
<td>1.05</td>
</tr>
<tr>
<td>mean $\mu$</td>
<td>4.42</td>
<td>4.53</td>
<td>2.25</td>
<td>2.10</td>
<td>1.02</td>
</tr>
<tr>
<td>std dev $\sigma$</td>
<td>0.17</td>
<td>0.10</td>
<td>0.10</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>$\sigma/\mu$</td>
<td>0.038</td>
<td>0.023</td>
<td>0.045</td>
<td>0.042</td>
<td>0.019</td>
</tr>
</tbody>
</table>
Reducing Horton’s laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto \text{sum of all stream lengths in a order } \Omega \text{ basin (assuming uniform drainage density)}$

- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{dd}$$

$$\propto \sum_{\omega=1}^{\Omega} R_n^{\Omega-\omega} \cdot \frac{n_\Omega}{n_\omega} \cdot \bar{s}_1 \cdot R_s^{\omega-1}$$

$$= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^\omega$$
Reducing Horton’s laws:

Continued ...

\[ a_\Omega \propto \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left( \frac{R_s}{R_n} \right)^\omega \]

\[ = \frac{R_n^\Omega}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^\Omega}{1 - (R_s/R_n)} \]

\[ \sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1 - (R_s/R_n)} \quad \text{as } \Omega \to \]

So, \( a_\Omega \) is growing like \( R_n^\Omega \) and therefore:

\[ R_n \equiv R_a \]
Reducing Horton’s laws:

Not quite:

- ... But this only a rough argument as Horton’s laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Problem set 1 question....
Equipartitioning:

Intriguing division of area:

- Observe: Combined area of basins of order $\omega$ independent of $\omega$.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:
  \[ R_n \equiv R_a \Rightarrow n_\omega \bar{a}_\omega = \text{const} \]
- Reason:
  \[ n_\omega \propto (R_n)^{-\omega} \]
  \[ \bar{a}_\omega \propto (R_a)^{\omega} \propto n_\omega^{-1} \]
Equipartitioning:

Some examples:
Scaling laws

The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga’s law describes detailed architecture: \( T_k = T_1 R_T^{k-1} \).
- We have connected Tokunaga’s and Horton’s laws
- Only two Horton laws are independent \( (R_n = R_a) \)
- Only two parameters are independent: \( (T_1, R_T) \leftrightarrow (R_n, R_s) \)
Scaling laws

A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network \( p \).
- Each point \( p \) is associated with a basin and a longest stream length
- **Q:** What is probability that the \( p \)'s drainage basin has area \( a \)? \( P(a) \propto a^{-\tau} \) for large \( a \)
- **Q:** What is probability that the longest stream from \( p \) has length \( \ell \)? \( P(\ell) \propto \ell^{-\gamma} \) for large \( \ell \)
- Roughly observed: \( 1.3 \lesssim \tau \lesssim 1.5 \) and \( 1.7 \lesssim \gamma \lesssim 2.0 \)
Scaling laws

Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf’s law)
  - Word frequency (Zipf’s law) [24]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]

- A big part of the story of complex systems

- Arise from mechanisms: growth, randomness, optimization, ...

- Our task is always to illuminate the mechanism...
Scaling laws

Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- Plan: Derive \( P(a) \propto a^{-\tau} \) and \( P(\ell) \propto \ell^{-\gamma} \) starting with Tokunaga/Horton story \([20, 1, 2]\)
- Let’s work on \( P(\ell) \)...
- Our first fudge: assume Horton’s laws hold throughout a basin of order \( \Omega \).
- (We know they deviate from strict laws for low \( \omega \) and high \( \omega \) but not too much.)
Scaling laws

Finding $\gamma$:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_>(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\text{max}}} P(\ell) d\ell$$

- Also known as the exceedance probability.
Scaling laws

Finding $\gamma$:

- The connection between $P(x)$ and $P_>(x)$ when $P(x)$ has a power law tail is simple:
- Given $P(\ell) \sim \ell^{-\gamma}$ large $\ell$ then for large enough $\ell_*$

\[
P_>(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\text{max}}} P(\ell) \, d\ell
\]

\[
\sim \int_{\ell=\ell_*}^{\ell_{\text{max}}} \ell^{-\gamma} \, d\ell
\]

\[
= \frac{\ell^{\gamma+1}}{-\gamma + 1} \bigg|_{\ell=\ell_*}^{\ell_{\text{max}}}
\]

\[
\propto \ell_*^{\gamma+1} \quad \text{for } \ell_{\text{max}} \gg \ell_*
\]
Scaling laws

Finding $\gamma$:

- **Aim:** determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution $\Delta$
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
- Approximate $P_{>}(\ell_*)$ as

$$P_{>}(\ell_*) = \frac{N_{>}(\ell_*; \Delta)}{N_{>}(0; \Delta)}.$$  

where $N_{>}(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$.

- Use Horton's law of stream segments:

$$s_\omega / s_{\omega-1} = R_s...$$
Scaling laws

Finding $\gamma$:

- Set $\ell_\ast = \ell_\omega$ for some $1 \ll \omega \ll \Omega$.

$$P_\ast(\ell_\omega) = \frac{N_\ast(\ell_\omega; \Delta)}{N_\ast(0; \Delta)} \approx \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}$$

- $\Delta$'s cancel
- Denominator is $a_\Omega \rho_{dd}$, a constant.
- So... using Horton's laws...

$$P_\ast(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \approx \sum_{\omega'=\omega+1}^{\Omega} \left(1.R_{n}^{\Omega-\omega'}(\bar{s}_1.R_{s}^{\omega'-1})\right)$$
Scaling laws

Finding $\gamma$:

- We are here:

$$P > (l_\omega) \propto \sum_{\omega' = \omega + 1}^{\Omega} (1 \cdot R_n^{\Omega - \omega'})(\bar{s}_1 \cdot R_s^{\omega' - 1})$$

- Cleaning up irrelevant constants:

$$P > (l_\omega) \propto \sum_{\omega' = \omega + 1}^{\Omega} \left( \frac{R_s}{R_n} \right)^{\omega'}$$

- Change summation order by substituting $\omega'' = \Omega - \omega'$. 

- Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)
Scaling laws

Finding $\gamma$:

$P_\gamma(\ell_\omega) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left( \frac{R_s}{R_n} \right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left( \frac{R_n}{R_s} \right)^{\omega''}$

Since $R_n < R_s$ and $1 \ll \omega \ll \Omega$,

$P_\gamma(\ell_\omega) \propto \left( \frac{R_n}{R_s} \right)^{\Omega-\omega} \propto \left( \frac{R_n}{R_s} \right)^{-\omega}$

again using $\sum_{i=0}^{n} a^n = (a^{i+1} - 1)/(a - 1)$
Scaling laws

Finding $\gamma$:

- Nearly there:

$$P_>(\ell, \omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

- Need to express right hand side in terms of $\ell, \omega$.
- Recall that $\ell \approx \ell_1 R_{\ell}^{\omega-1}$.
- 

$$\ell \propto R_{\ell}^{\omega} = R_s^\omega = e^{\omega \ln R_s}$$
Scaling laws

Finding $\gamma$:

- Therefore:

\[ P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = (e^{\omega \ln R_s})^{-\ln(R_n/R_s) / \ln(R_s)} \]

- \[ \propto \ell_\omega - \ln(R_n/R_s) / \ln R_s \]

- \[ = \ell_\omega - (\ln R_n - \ln R_s) / \ln R_s \]

- \[ = \ell_\omega - \ln R_n / \ln R_s + 1 \]

- \[ = \ell_\omega - \gamma + 1 \]
Scaling laws

Finding $\gamma$:

- And so we have:
  \[ \gamma = \ln R_n / \ln R_s \]

- Proceeding in a similar fashion, we can show
  \[ \tau = 2 - \ln R_s / \ln R_n = 2 - 1 / \gamma \]

- Such connections between exponents are called scaling relations

- Let’s connect to one last relationship: Hack’s law
Scaling laws

Hack’s law: \cite{6}

\[ l \propto a^h \]

- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
- Use Horton laws to connect $h$ to Horton ratios:

\[ l_\omega \propto R_s^\omega \text{ and } a_\omega \propto R_n^\omega \]

- Observe:

\[ l_\omega \propto e^{\omega \ln R_s} \propto \left( e^{\omega \ln R_n} \right)^{\ln R_s / \ln R_n} \]

\[ \propto (R_n^\omega)^{\ln R_s / \ln R_n} = a_\omega^{\ln R_s / \ln R_n} \Rightarrow h = \ln R_s / \ln R_n \]
Connecting exponents

Only 3 parameters are independent: e.g., take \( d \), \( R_n \), and \( R_s \)

<table>
<thead>
<tr>
<th>relation:</th>
<th>scaling relation/parameter: [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell \sim L^d )</td>
<td>( d )</td>
</tr>
<tr>
<td>( T_k = T_1(R_T)^{k-1} )</td>
<td>( T_1 = R_n - R_s - 2 + 2R_s/R_n )</td>
</tr>
<tr>
<td>( n_\omega/n_{\omega+1} = R_n )</td>
<td>( R_n )</td>
</tr>
<tr>
<td>( a_{\omega+1}/a_\omega = R_a )</td>
<td>( R_a = R_n )</td>
</tr>
<tr>
<td>( \ell_{\omega+1}/\ell_\omega = R_\ell )</td>
<td>( R_\ell = R_s )</td>
</tr>
<tr>
<td>( \ell \sim a^h )</td>
<td>( h = \log R_s/\log R_n )</td>
</tr>
<tr>
<td>( a \sim L^D )</td>
<td>( D = d/h )</td>
</tr>
<tr>
<td>( L_\perp \sim L^H )</td>
<td>( H = d/h - 1 )</td>
</tr>
<tr>
<td>( P(a) \sim a^{-\tau} )</td>
<td>( \tau = 2 - h )</td>
</tr>
<tr>
<td>( P(\ell) \sim \ell^{-\gamma} )</td>
<td>( \gamma = 1/h )</td>
</tr>
<tr>
<td>( \Lambda \sim a^\beta )</td>
<td>( \beta = 1 + h )</td>
</tr>
<tr>
<td>( \lambda \sim L^\varphi )</td>
<td>( \varphi = d )</td>
</tr>
</tbody>
</table>
Equipartitioning reexamined:

Recall this story:

1. Mississippi basin partitioning
2. Amazon basin partitioning
3. Nile basin partitioning

References
Equipartitioning

- What about
  \[ P(a) \sim a^{-\tau} \] ?
- Since \( \tau > 1 \), suggests no equipartitioning:
  \[ aP(a) \sim a^{-\tau+1} \neq \text{const} \]
- \( P(a) \) overcounts basins within basins...
- while stream ordering separates basins...
Moving beyond the mean:

- Both Horton’s laws and Tokunaga’s law relate average properties, e.g.,

\[
\frac{\tilde{s}_\omega}{\tilde{s}_{\omega-1}} = R_s
\]

- Natural generalization to consideration relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness...
A toy model—Scheidegger’s model

Directed random networks \[^{[12, 13]}\]

\[
P(\downarrow) = P(\leftarrow) = 1/2
\]

- Flow is directed downwards
- Useful and interesting test case—more later...
Generalizing Horton’s laws

\[ \ell_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell / R_\ell^\omega) \]

\[ a_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a / R_n^\omega) \]

- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order
Generalizing Horton’s laws

How well does overall basin fit internal pattern?

- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.
Generalizing Horton’s laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in $10^3$ km):

<table>
<thead>
<tr>
<th>basin</th>
<th>$\ell_\Omega$</th>
<th>$\bar{\ell}_\Omega$</th>
<th>$\sigma_\ell$</th>
<th>$\ell/\ell_\Omega$</th>
<th>$\sigma_\ell/\ell_\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mississippi</td>
<td>4.92</td>
<td>11.10</td>
<td>5.60</td>
<td>0.44</td>
<td>0.51</td>
</tr>
<tr>
<td>Amazon</td>
<td>5.75</td>
<td>9.18</td>
<td>6.85</td>
<td>0.63</td>
<td>0.75</td>
</tr>
<tr>
<td>Nile</td>
<td>6.49</td>
<td>2.66</td>
<td>2.20</td>
<td>2.44</td>
<td>0.83</td>
</tr>
<tr>
<td>Congo</td>
<td>5.07</td>
<td>10.13</td>
<td>5.75</td>
<td>0.50</td>
<td>0.57</td>
</tr>
<tr>
<td>Kansas</td>
<td>1.07</td>
<td>2.37</td>
<td>1.74</td>
<td>0.45</td>
<td>0.73</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$\bar{a}_\Omega$</th>
<th>$\sigma_a$</th>
<th>$a/\bar{a}_\Omega$</th>
<th>$\sigma_a/\bar{a}_\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mississippi</td>
<td>2.74</td>
<td>7.55</td>
<td>5.58</td>
<td>0.36</td>
<td>0.74</td>
</tr>
<tr>
<td>Amazon</td>
<td>5.40</td>
<td>9.07</td>
<td>8.04</td>
<td>0.60</td>
<td>0.89</td>
</tr>
<tr>
<td>Nile</td>
<td>3.08</td>
<td>0.96</td>
<td>0.79</td>
<td>3.19</td>
<td>0.82</td>
</tr>
<tr>
<td>Congo</td>
<td>3.70</td>
<td>10.09</td>
<td>8.28</td>
<td>0.37</td>
<td>0.82</td>
</tr>
<tr>
<td>Kansas</td>
<td>0.14</td>
<td>0.49</td>
<td>0.42</td>
<td>0.28</td>
<td>0.86</td>
</tr>
</tbody>
</table>
Combining stream segments distributions:

- Stream segments sum to give main stream lengths

\[ \ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu \]

- \( P(\ell_\omega) \) is a convolution of distributions for the \( s_\omega \)
Generalizing Horton’s laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell | \omega) = N(s | 1) \ast N(s | 2) \ast \cdots \ast N(s | \omega)$$

$$N(s | \omega) = \frac{1}{R_n^\omega R_\ell^\omega} F(s / R_\ell^\omega)$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.
Generalizing Horton’s laws

Next level up: Main stream length distributions must combine to give overall distribution for stream length

$P(\ell) \sim \ell^{-\gamma}$

Another round of convolutions [3]

Interesting...
Generalizing Horton’s laws

Number and area distributions for the Scheidegger model $P(n_{1,6})$ versus $P(a_6)$. 
Generalizing Tokunaga’s law

Scheidegger:

- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using $R_s$
Generalizing Tokunaga’s law

Mississippi:

- Same data collapse for Mississippi...
Generalizing Tokunaga’s law

So

\[ P(T_{\mu, \nu}) = (R_s)^{\mu-\nu-1} P_t \left[ \frac{T_{\mu, \nu}}{(R_s)^{\mu-\nu-1}} \right] \]

where

\[ P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t} \]

\[ P(s_{\mu}) \leftrightarrow P(T_{\mu, \nu}) \]

- Exponentials arise from randomness.
- Look at joint probability \( P(s_{\mu}, T_{\mu, \nu}) \).
Generalizing Tokunaga’s law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments
Generalizing Tokunaga’s law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order $\mu$ stream segment terminating is constant:
  $$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1}\xi_s$$
- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- $\Rightarrow$ random spatial distribution of stream segments
Generalizing Tokunaga’s law

Joint distribution for generalized version of Tokunaga’s law:

\[ P(s_{\mu}, T_{\mu,\nu}) = \tilde{\rho}_\mu \left( \frac{s_{\mu} - 1}{T_{\mu,\nu}} \right) \rho^T_{\nu,\nu} (1 - p_\nu - \tilde{\rho}_\mu)^{s_{\mu} - T_{\mu,\nu} - 1} \]

where

- \( p_\nu = \) probability of absorbing an order \( \nu \) side stream
- \( \tilde{\rho}_\mu = \) probability of an order \( \mu \) stream terminating

Approximation: depends on distance units of \( s_{\mu} \)

In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.
Generalizing Tokunaga’s law

Now deal with thing:

\[ P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \left( \frac{s_\mu - 1}{T_{\mu,\nu}} \right) p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1} \]

Set \((x, y) = (s_\mu, T_{\mu,\nu})\) and \(q = 1 - p_\nu - \tilde{p}_\mu\), approximate liberally.

Obtain

\[ P(x, y) = N x^{-1/2} [F(y/x)]^x \]

where

\[ F(\nu) = \left( \frac{1 - \nu}{q} \right)^{-(1-\nu)} \left( \frac{\nu}{p} \right)^{-\nu}. \]
Generalizing Tokunaga’s law

Checking form of $P(s_{\mu}, T_{\mu,\nu})$ works:

Scheidegger:

(a) $v = \frac{T_{\mu,\nu}}{l_{\mu}^{(s)}}$

(b) $P(v | l_{\mu}^{(s)})$

$F(v | l_{\mu}^{(s)})$
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:
Generalizing Tokunaga’s law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Mississippi:
Random subnetworks on a Bethe lattice \cite{15}

- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of “Statistical inevitability” of river network statistics \cite{8}
- But Bethe lattices unconnected with surfaces.
- In fact, Bethe lattices $\sim$ infinite dimensional spaces (oops).
- So let’s move on...
Scheidegger’s model

Directed random networks $^{[12, 13]}$

$P(\downarrow) = P(\uparrow) = 1/2$

- Functional form of all scaling laws exhibited but exponents differ from real world $^{[18, 19, 17]}$
A toy model—Scheidegger’s model

Random walk basins:

- Boundaries of basins are random walks
Scheidegger’s model

Increasing partition of N=64
Scheidegger’s model

Prob for first return of a random walk in (1+1) dimensions:

\[ P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}. \]

and so \( P(\ell) \propto \ell^{-3/2} \).

Typical area for a walk of length \( n \) is \( \propto n^{3/2} \):

\[ \ell \propto a^{2/3}. \]

Find \( \tau = 4/3, \ h = 2/3, \ \gamma = 3/2, \ d = 1 \).

Note \( \tau = 2 - h \) and \( \gamma = 1/h \).

\( R_n \) and \( R_\ell \) have not been derived analytically.
Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [11]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int d\vec{r} \ (\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^\gamma$$

- Landscapes obtained numerically give exponents near that of real networks.

- But: numerical method used matters.

- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [9]
Theoretical networks

Summary of universality classes:

<table>
<thead>
<tr>
<th>network</th>
<th>h</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-convergent flow</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Directed random</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>Undirected random</td>
<td>5/8</td>
<td>5/4</td>
</tr>
<tr>
<td>Self-similar</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>OCN’s (I)</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>OCN’s (II)</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td>OCN’s (III)</td>
<td>3/5</td>
<td>1</td>
</tr>
<tr>
<td>Real rivers</td>
<td>0.5–0.7</td>
<td>1.0–1.2</td>
</tr>
</tbody>
</table>

\[ h \Rightarrow \ell \propto a^h \text{ (Hack's law)}. \]
\[ d \Rightarrow \ell \propto L^d_{||} \text{ (stream self-affinity)}. \]
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