Singular Value Decomposition
Linear Algebra, Course 124A, Fall, 2007

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Approximating matrices with SVD

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The basic idea

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Bonus Puzzle

Colbert on Equations
Image approximation (80x60)

- Idea: use SVD to approximate images.
- Interpret elements of matrix $A$ as color values of an image.
- Truncate series SVD representation of $A$:
  \[
  A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T
  \]
- Use fact that $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$.
- For color: approximate 3 matrices (RGB).
Image approximation (80x60)

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Image approximation (80x60)

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\[ A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{2} \sigma_i \hat{u}_i \hat{v}_i^T \]
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\[ A = \sum_{i=1}^{3} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{4} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

$$A = \sum_{i=1}^{5} \sigma_i \hat{u}_i \hat{v}_i^T$$
Image approximation (80x60)

\[
A = \sum_{i=1}^{6} \sigma_i \hat{u}_i \hat{v}_i^T
\]
Image approximation (80x60)

\[ A = \sum_{i=1}^{7} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{8} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{9} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{10} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{20} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Image approximation (80x60)

\[ A = \sum_{i=1}^{30} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{40} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{50} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (80x60)

\[ A = \sum_{i=1}^{60} \sigma_i \hat{u}_i \hat{v}_i^T \]
Decay of sigma values: Einstein

\[ \sigma_k \]

\[ k \]

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\[ A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Image approximation (480x640)

\[ A = \sum_{i=1}^{2} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{3} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

$$A = \sum_{i=1}^{4} \sigma_i \hat{u}_i \hat{v}_i^T$$
Image approximation (480x640)

\[ A = \sum_{i=1}^{5} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{6} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Image approximation (480x640)

\[ A = \sum_{i=1}^{7} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{8} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{9} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{10} \sigma_i \hat{u}_i \hat{v}_i^T \]
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\[ A = \sum_{i=1}^{60} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{100} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Image approximation (480x640)

\[ A = \sum_{i=1}^{200} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{320} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Image approximation (480x640)

\[ A = \sum_{i=1}^{480} \sigma_i \hat{u}_i \hat{v}_i^T \]
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\[ A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{2} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{3} \sigma_i \hat{u}_i \hat{v}_i^T \]
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\[ A = \sum_{i=1}^{4} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{5} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

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A = \sum_{i=1}^{6} \sigma_i \hat{u}_i \hat{v}_i^T
\]
Image approximation (480x640)

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$$A = \sum_{i=1}^{40} \sigma_i \hat{u}_i \hat{v}_i^T$$
Image approximation (480x640)

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Image approximation (480x640)

\[ A = \sum_{i=1}^{200} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{320} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

\[ A = \sum_{i=1}^{480} \sigma_i \hat{u}_i \hat{v}_i^T \]
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Image approximation (480x640)

\[ A = \sum_{i=1}^{20} \sigma_i \hat{u}_i \hat{v}_i^T \]
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\[ A = \sum_{i=1}^{60} \sigma_i \hat{u}_i \hat{v}_i^T \]
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A = \sum_{i=1}^{100} \sigma_i \hat{u}_i \hat{v}_i^T
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\[ A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^T \]
Image approximation (480x640)

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Image approximation (480x640)

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Colbert on Equations
The truth about mathematics

The Colbert Report on Math (February 2, 2007)