Mechanisms for Generating Power-Law Size Distributions I

Principles of Complex Systems
CSYS/MATH 300, Spring, 2013

Prof. Peter Dodds
@peterdodds

Department of Mathematics & Statistics | Center for Complex Systems | Vermont Advanced Computing Center | University of Vermont

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Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks. (⊞)

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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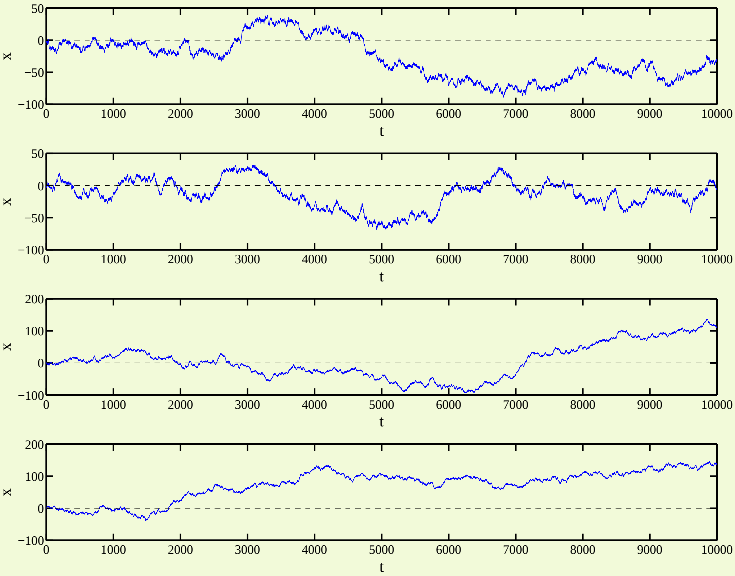
A few random random walks:



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

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Variances sum: (⊞)*

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\boxed{\sigma = t^{1/2}}$$

- A non-trivial scaling law arises out of additive aggregation or accumulation.

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Great moments in Televised Random Walks:

Plinko! (⊞) from the Price is Right.

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 2 (⊞)

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

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How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \ldots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 2 (⊞)

- The whole is different from the parts. #nutritious
- See also: Stable Distributions (⊞)

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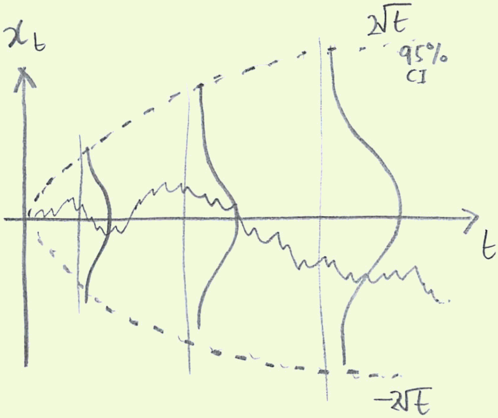
Universality (⊞) is also not left-handed:



- This is Diffusion (⊞): the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.

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Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞!

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Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞!

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See Feller, Intro to Probability Theory, Volume I [3]

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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

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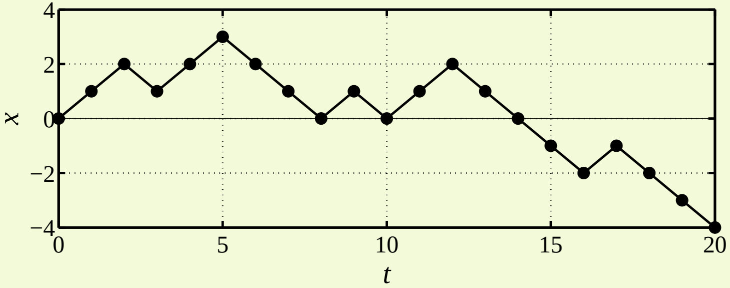

- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\rm fr}(2n)$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

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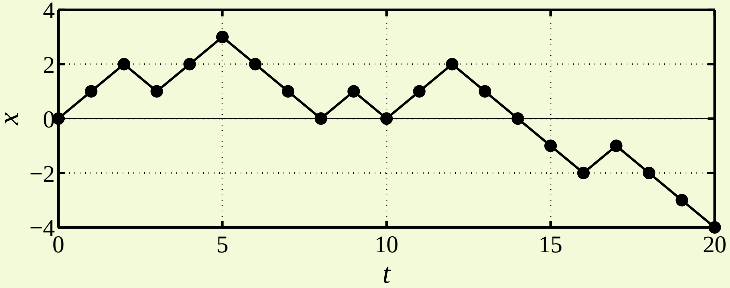

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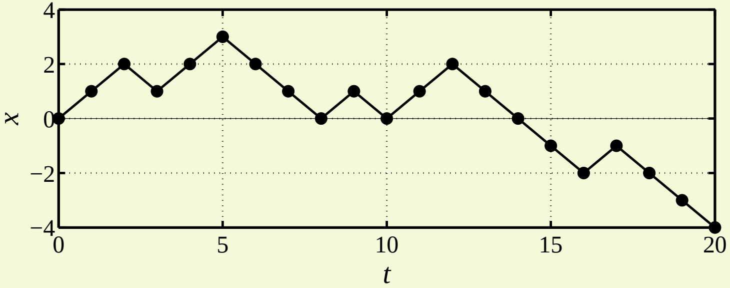

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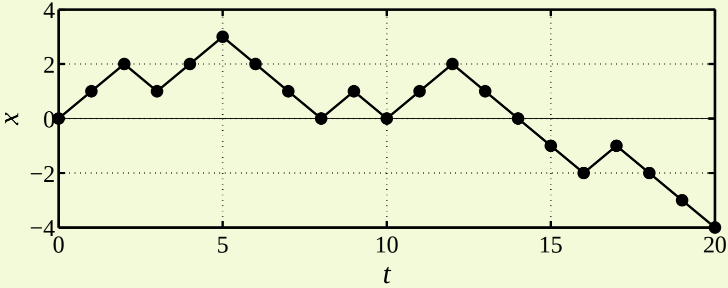

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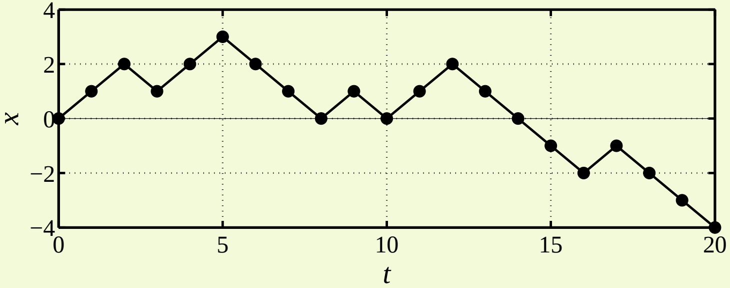

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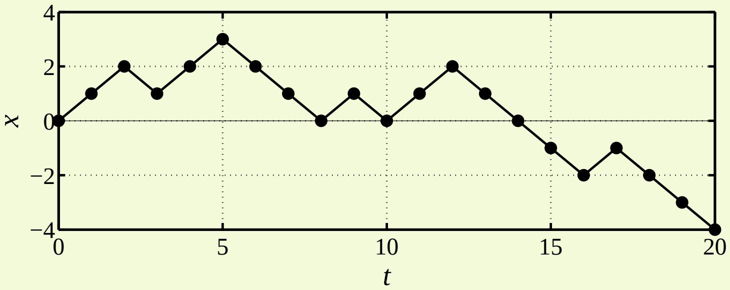

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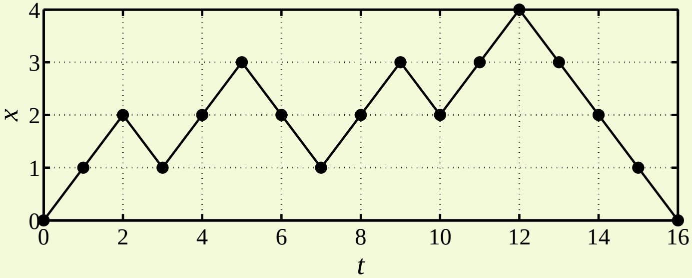

- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\mathrm{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.

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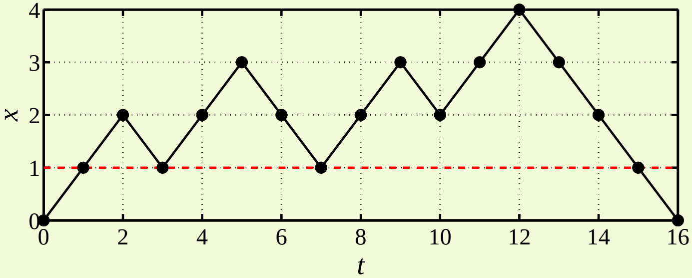

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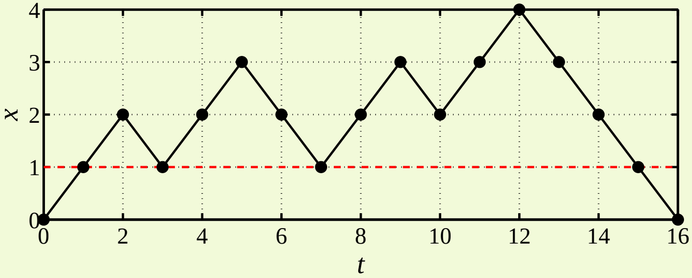

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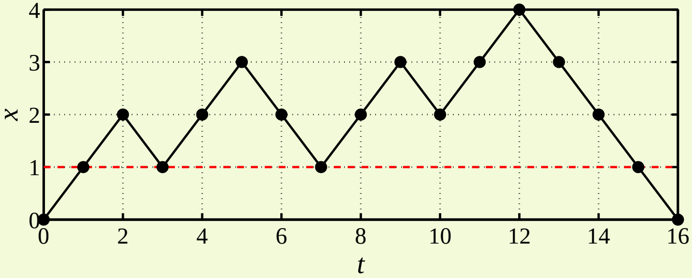

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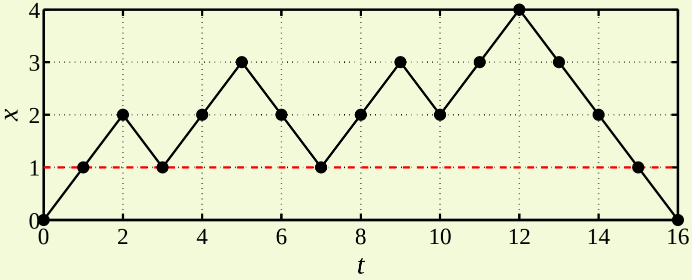

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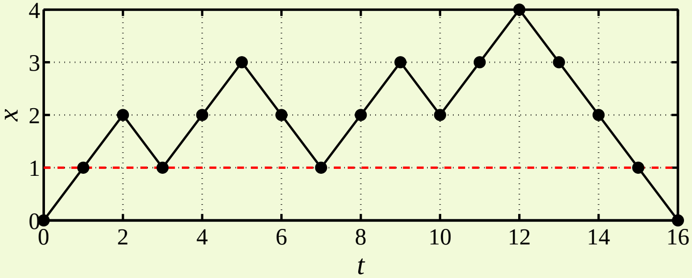

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Counting first returns:

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Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

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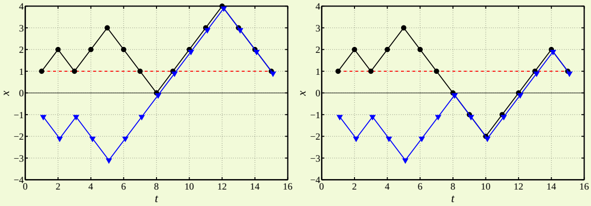

Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

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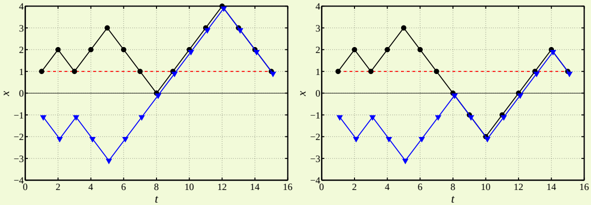

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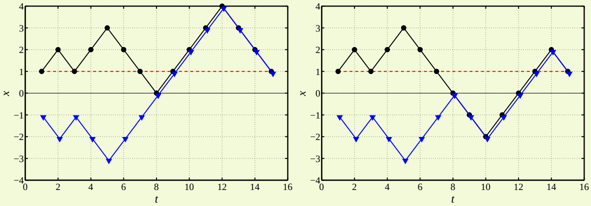

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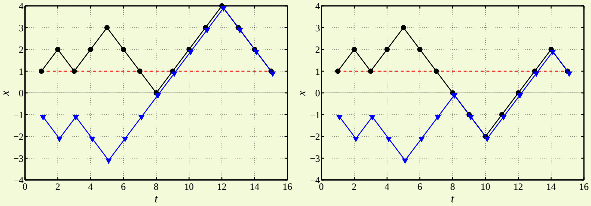

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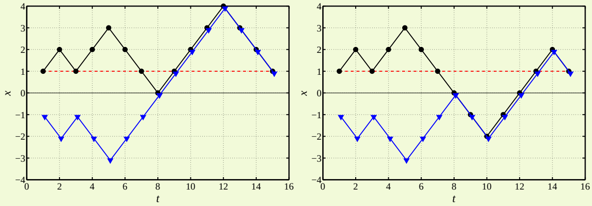
- So $N_{\text{first return}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$

Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
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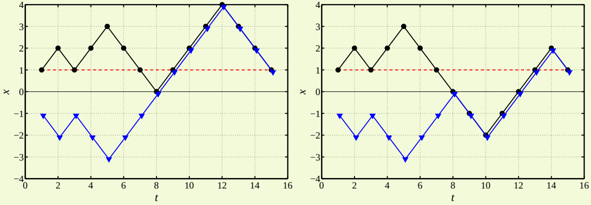

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Probability of first return:

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Insert question from assignment 2 (⊞) :

- Find

$$\boxed{N_{\mathrm{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.}$$

- Normalized number of paths gives probability.
- Total number of possible paths $= 2^{2n}$.
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$$P_{\mathrm{fr}}(2n) = \frac{1}{2^{2n}} N_{\mathrm{fr}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}}(2n)^{-3/2} \propto t^{-3/2}.$$

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- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions (⊞):

- Walker in $d = 2$ dimensions must also return
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On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present:
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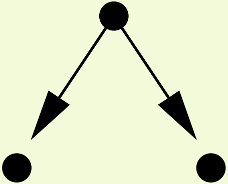

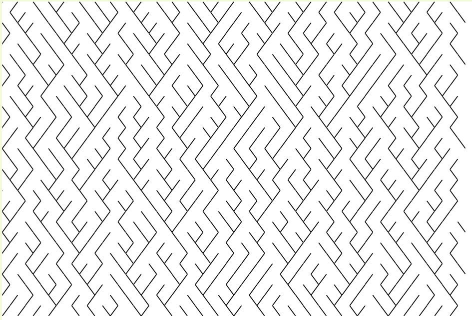

- Random directed network on triangular lattice.
- Toy model of real networks.
- 'Flow' is southeast or southwest with equal probability.

Scheidegger networks

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- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

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Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
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Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
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Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [4]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (later: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

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$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

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Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
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Other First Returns or First Passage Times:

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Failure:

- A very simple model of failure/death:[10]
- x_t = entity's 'health' at time t
- Start with $x_0 > 0$.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

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More than randomness

- Can generalize to Fractional Random Walks [6, 7, 5]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [5]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

$$\sigma \sim t^{\alpha}$$

- Extensive memory of path now matters...

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Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$ $= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
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2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$
  $= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$

- Often easier to do by hand...

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Variable Transformation

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General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
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$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

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Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If $P_x(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then

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Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.

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- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

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Matter is concentrated in stars: [9]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3 \mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

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Transformation:

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Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

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Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (⊞)...
- Don't do this!!! (slap, slap)
- We need mechanisms!

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