Mechanisms for Generating Power-Law Size Distributions I

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Outline

Mechanisms:

A powerful story in the rise of complexity:

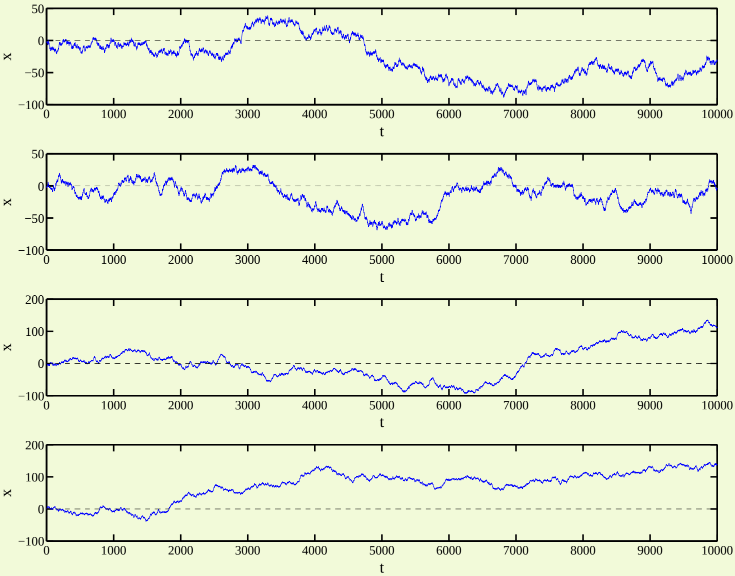
- structure arises out of randomness.
- Exhibit A: Random walks. (⊞)

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

A few random random walks:



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Variances sum: (⊞)*

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\boxed{\sigma = t^{1/2}}$$

- A non-trivial scaling law arises out of additive aggregation or accumulation.

Great moments in Televised Random Walks:

Plinko! (⊞) from the Price is Right.

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 2 (⊞)

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \ldots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

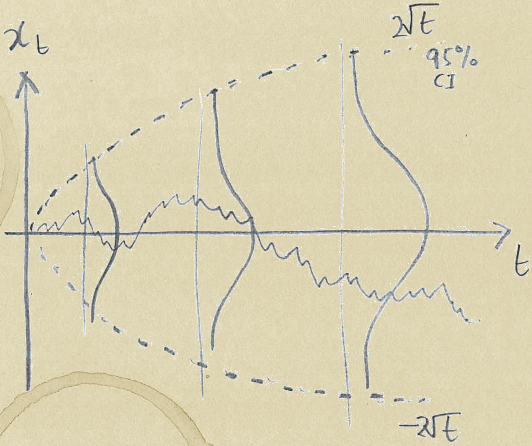
- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 2 (⊞)

- The whole is different from the parts. #nutritious
- See also: Stable Distributions (⊞)

Universality (⊞) is also not left-handed:



- This is Diffusion (⊞): the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞!

See Feller, Intro to Probability Theory, Volume I [3]

Random walks #crazytownbananapants

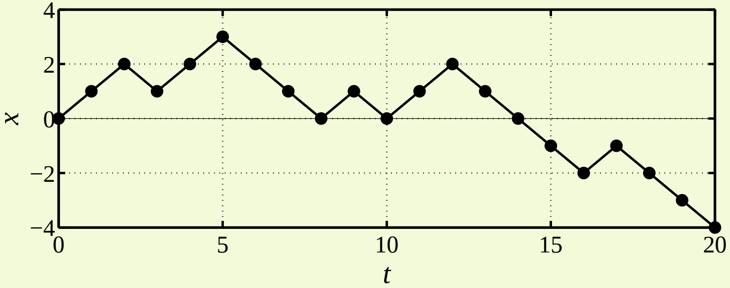
The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

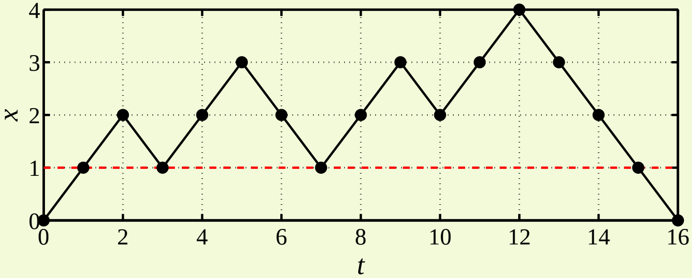
Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

For random walks in 1-d:



- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\rm fr}(2n)$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.



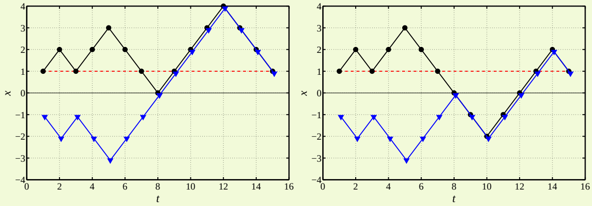
- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n - 1,\ \text{and}\ x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x{=}1$ that hits 0, there is a unique matching path starting at $x{=}-1$.
- Matching path first mirrors and then tracks after first reaching $x{=}0$.
- # of t-step paths starting and ending at $x{=}1$ and hitting $x{=}0$ at least once
  = # of t-step paths starting at $x{=}-1$ and ending at $x{=}1$ = $N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

Probability of first return:

Insert question from assignment 2 (⊞) :

- Find

$$\boxed{N_{\rm fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.}$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\rm fr}(2n) = \frac{1}{2^{2n}} N_{\rm fr}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}}(2n)^{-3/2} \propto t^{-3/2}.$$

First Returns

- $$P(t) \propto t^{-3/2},\ \gamma = 3/2$$
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions (⊞):

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

Random walks

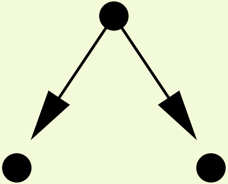
On finite spaces:

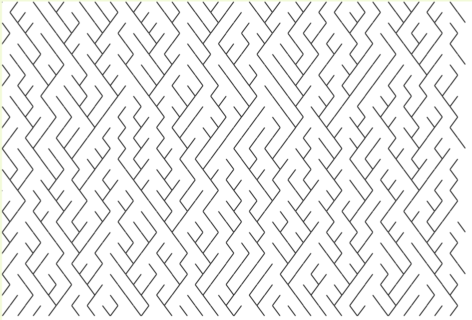
- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present:
  walkers traverse edges with equal frequency.
  #totallygroovy

Scheidegger Networks [8, 2]





- Random directed network on triangular lattice.
- Toy model of real networks.
- 'Flow' is southeast or southwest with equal probability.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- **Pr**(basin area $= a)\mathrm{d}a$
  $= \textbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [4]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (later: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Given

$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
- $$\boxed{\tau = 1 + h(\gamma - 1)}$$
- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to:[1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality (⊞) class with independent exponents.

Other First Returns or First Passage Times:

Failure:

- A very simple model of failure/death: [10]
- x_t = entity's 'health' at time t
- Start with $x_0 > 0$.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

More than randomness

- Can generalize to Fractional Random Walks [6, 7, 5]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [5]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

$$\sigma \sim t^{\alpha}$$

  $\alpha = 1/2$ — diffusive
  $\alpha > 1/2$ — superdiffusive
  $\alpha < 1/2$ — subdiffusive
- Extensive memory of path now matters...

Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$ $= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...

General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$
  $$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$

Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

$$P_y(y)\mathrm{d}y = P_x \overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)} \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \to$ non-zero constant as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty.$$

- If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.

Gravity

- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Matter is concentrated in stars: [9]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3 \mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$
- $$\propto P_r(F^{-1/2})F^{-3/2}\mathrm{d}F$$
- $$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$
- $$= F^{-1-3/2}\mathrm{d}F$$
- $$= F^{-5/2}\mathrm{d}F\,.$$

Gravity:

- $$P_F(F) = F^{-5/2}\mathrm{d}F$$
  $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (⊞)...
- Don't do this!!! (slap, slap)
- We need mechanisms!

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