Optical precursor fields in nonlinear pulse dynamics

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Abstract: Under certain conditions, ultrashort pulse dynamics in a linear dispersive medium with absorption result in the appearance of optical precursors that dominate the pulse evolution for large propagation distances as the peak amplitude in the initial pulse spectrum decays exponentially. The effects of a nonlinear medium response on this precursor formation is considered using the split-step Fourier method. Comparison of the nonlinear pulse evolution when the full dispersion is used to that when a quadratic Taylor series approximation of the wave number is used shows that the group velocity approximation misses the precursor fields entirely.

OCIS codes: (260.2030) Dispersion; (320.5550) Pulses.

References and links
1. Introduction

The asymptotic description of ultrawideband dispersive pulse propagation in a Lorentz model dielectric has its origin in the classic research by Sommerfeld [1] and Brillouin [2, 3] based on Debye’s steepest descent method [4]. This seminal work established the physical phenomena of forerunners, or precursor fields [5], that were originally associated with the evolution of a Heaviside step-function signal. This description is derived from the exact Fourier-Laplace integral representation of the propagated linearly polarized plane wave field [6, 7, 8]

\[ E(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega, \tag{1} \]

for \( z \geq 0 \) with fixed \( a \) larger than the abscissa of absolute convergence [5, 6, 7, 8] for the spectrum \( \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt \) of the initial plane wave pulse \( E(0,t) = f(t) \) at \( z = 0 \). The temporal Fourier spectrum \( \tilde{E}(z,\omega) \) of the optical wave field \( E(z,t) \) satisfies the Helmholtz equation

\[ (\nabla^2 + \tilde{k}^2(\omega)) \tilde{E}(z,\omega) = 0 \tag{2} \]

with complex wave number \( \tilde{k}(\omega) = \beta(\omega) + i\alpha(\omega) = (\omega/c)n(\omega) \) in the temporally dispersive medium with complex index of refraction \( n(\omega) = n_r(\omega) + in_i(\omega) \) whose real \( n_r(\omega) = \Re \{ n(\omega) \} \) and imaginary \( n_i(\omega) = \Im \{ n(\omega) \} \) parts are related through the Kramers-Kronig relations [7]. Here \( \beta(\omega) = \Re \{ \tilde{k}(\omega) \} \) is the propagation (or phase) factor and \( \alpha(\omega) = \Im \{ \tilde{k}(\omega) \} \) the attenuation factor for plane wave propagation in the dispersive attenuative medium. In addition,

\[ \phi(\omega,\theta) = i\frac{c}{z} [\tilde{k}(\omega)z - \omega t] = i\omega [n(\omega) - \theta] \tag{3} \]

with \( \theta \equiv ct/z \) a nondimensional space-time parameter defined for all \( z > 0 \).

Based upon this exact integral representation, the modern asymptotic theory [9, 10] provides a uniform asymptotic description of dispersive pulse dynamics in a single-resonance Lorentz model dielectric with causal, complex index of refraction given by [7]

\[ n(\omega) = \left( 1 - \frac{\omega_0^2}{\omega^2 - \omega_p^2 + 2i\delta \omega} \right)^{1/2}, \tag{4} \]

with resonance frequency \( \omega_0 \), damping \( \delta \), and plasma frequency \( \omega_p \). The absorption band, which corresponds to the region of anomalous dispersion, then extends from \( \sim (\omega_0^2 - \delta^2)^{1/2} \) to \( \sim (\omega_0^2 - \delta^2)^{1/2} \), where \( \omega_1 \equiv (\omega_0^2 + \omega_p^2)^{1/2} \). Sommerfeld’s relativistic causality theorem then holds [1, 8], stating that if \( E(0,t) = 0 \) for all \( t < 0 \), then \( E(z,t) = 0 \) for all \( \theta < 1 \) with \( z > 0 \).

For \( \theta \geq 1 \), the asymptotic theory [2, 3, 6, 9] shows that the propagated wave field of an ultrawideband signal in a single-resonance Lorentz model dielectric is due to two independent sets of saddle points and pole contributions from the initial pulse spectrum so that it may then be expressed either as a superposition of component fields as

\[ E(z,t) = E_S(z,t) + E_B(z,t) + E_c(z,t), \tag{5} \]

or as a linear combination of fields of this form [8]. Here \( E_S(z,t) \) is the first forerunner [1, 2, 3] or Sommerfeld precursor due to a pair of distant saddle points describing the high-frequency \( (|\alpha| \geq \alpha_h) \) response of the dispersive medium, \( E_B(z,t) \) is the second forerunner [2, 3] or Brillouin precursor due to a pair of near saddle points describing the low-frequency \( (|\alpha| \leq \alpha_h) \) response, and \( E_c(z,t) \) is due to the residue of the pole contribution [8, 10] describing the signal contribution (if any). The observed pulse distortion for a Heaviside step function signal is then
seen to be due to both the precursor fields and their interference with the signal contribution. For a gaussian envelope pulse, the dynamical pulse evolution is comprised of just the precursor fields \([8, 11]\) because its spectrum \(\tilde{E}_G(0, \omega)\) is an entire function of complex \(\omega\). Of particular importance here is that the peak amplitude of the Brillouin precursor \(E_B(z, t)\) experiences zero exponential decay with propagation distance \(z > 0\), decreasing algebraically as \(z^{-1/2}\) in the dispersive, absorptive medium for \(\delta > 0\) bounded away from zero \([6, 8, 9, 10]\). Part of the purpose of this paper is to show that the precursor field components of an ultrawideband pulse persist when nonlinear effects are included.

By comparison, the group velocity description of dispersive pulse propagation was formulated by Havelock \([12, 13]\) based upon Kelvin’s stationary phase method \([14]\). In this approach, the wave number \(k(\omega)\), assumed there to be real-valued, is expanded in a Taylor series about some wave number value \(k_0\) that the spectrum of the wave group is clustered about, where \([13]\) “the range of integration is supposed to be small and the amplitude, phase and velocity of the members of the group are assumed to be continuous, slowly varying, functions” of the wave number. Notice that Havelock’s group method \([12, 13]\) is a significant departure from Kelvin’s stationary phase method \([14]\), where \(k_0\) is the stationary phase point of the wavenumber \(k(\omega)\). This apparently subtle change in the value of \(k_0\) results in significant consequences on the accuracy of the resulting group velocity description based on this method \([8, 15, 16]\).

![Propagation Factor Comparison](image1)

![Attenuation Coefficient Comparison](image2)

Fig. 1. Exact (blue curves) and quadratic Taylor series approximation (green curves) of the scaled propagation factor \((c/\omega)\beta(\omega)\) (upper plot) and the attenuation coefficient \(\alpha(\omega)\) of the complex wave number for a single resonance Lorentz model dielectric with \(\omega_0 = 2.4 \times 10^{15} r/s, \delta = 6.0 \times 10^{13} r/s, \omega_p = 3.05 \times 10^{12} r/s\).
The accuracy of the quadratic Taylor series approximation of the complex wave number

\[ \tilde{k}(\omega) \approx \tilde{k}(\omega_0) + \tilde{k}'(\omega_0)(\omega - \omega_0) + \frac{1}{2}\tilde{k}''(\omega_0)(\omega - \omega_0)^2, \]

where \( \tilde{k}'(\omega) = \frac{\partial \tilde{k}(\omega)}{\partial \omega} \) (the inverse of the group velocity) and \( \tilde{k}''(\omega) = \frac{\partial^2 \tilde{k}(\omega)}{\partial \omega^2} \) (the so-called group velocity dispersion), is illustrated in Fig. 1 with \( \omega_0 = 1.0 \times 10^{15}/s \) for a moderately dispersive resonance line at \( \omega_0 = 2.4 \times 10^{15}/s \) with \( \omega_p/\omega_0 \approx 0.00127 \) and \( \delta/\omega_0 = 0.025 \), the upper plot describing the scaled real part \( (c/\omega)|\beta(\omega) - \beta(\omega_0)| \) and the lower plot the imaginary part \( \alpha(\omega) \) of \( \tilde{k}(\omega) \). Although \( \beta(\omega) \) is accurately modeled over a finite frequency interval below resonance (because \( \omega_0 < \omega_0 \) in this case, diverging from the actual behavior both when \( \omega \) increases sufficiently above or below the medium resonance frequency \( \omega_0 \), \( \alpha(\omega) \) is overestimated for all frequencies above and below the carrier frequency \( \omega_0 \). Because of this, the frequency dependence of \( \alpha(\omega) \) is typically neglected in the group velocity approximation where \( \alpha(\omega) \approx \alpha(\omega_0) \). The inclusion of higher-order terms in the Taylor series approximation of \( \tilde{k}(\omega) \) beyond the quadratic dispersion approximation given in Eq. (6) has been shown [8, 15, 16] to yield no improvement in accuracy, and hence, is meaningless.

As a consequence of these error sources, which are fundamental to the group method, the accuracy of the group velocity approximation rapidly decreases in the ultrawideband signal, ultrashort pulse limit as the pulse rise- or fall-time decreases below the characteristic relaxation time of the medium resonance [8, 15, 16]. This result is now extended to the nonlinear regime where the relative importance of the precursor fields has so far been neglected.

2. Formulation of the nonlinear problem

The formulation of the nonlinear dispersive pulse propagation problem begins with the inhomogeneous scalar wave equation (see §11.3 of [8] and §1.3 of [17])

\[ \nabla^2 E(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 E(\mathbf{r}, t)}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P(\mathbf{r}, t)}{\partial t^2} \]

for a linearly polarized pulse in a simple dispersive medium characterized by the constitutive relation \( D(\mathbf{r}, t) = E(\mathbf{r}, t) + 4\pi P(\mathbf{r}, t) \). With substitution of the complex phasor representation \( E_{\omega_0}(\mathbf{r}, t) \equiv A(\mathbf{r}, t)e^{i\phi(\mathbf{r}, t)e^{-\alpha_0 t}} \), where \( E(\mathbf{r}, t) = \Re\{E_{\omega_0}(\mathbf{r}, t)\} \), the linear material polarization response is found as [8] \( P_{\omega_0}(\mathbf{r}, t) = \chi_e(\omega_0)A(\mathbf{r}, t)e^{i\phi(\mathbf{r}, t)e^{-\alpha_0 t}} \), where \( P(\mathbf{r}, t) = \Re\{P_{\omega_0}(\mathbf{r}, t)\} \)

and \( \chi_e(\omega) \) denotes the electric susceptibility, and the wave equation (7) becomes

\[ \nabla^2 E_{\omega_0}(\mathbf{r}, t) - \frac{n^2(\omega_0)}{c^2} \frac{\partial^2 E_{\omega_0}(\mathbf{r}, t)}{\partial t^2} = 0. \]

(8)

With the inclusion of a cubic nonlinearity [17], this wave equation may be generalized as

\[ \nabla^2 E_{\omega_0}(\mathbf{r}, t) - \frac{n^2(\omega_0)}{c^2} \frac{\partial^2 E_{\omega_0}(\mathbf{r}, t)}{\partial t^2} + \gamma|E_{\omega_0}(\mathbf{r}, t)|^2 E_{\omega_0}(\mathbf{r}, t) = 0, \]

(9)

where \( \gamma \) is the nonlinear-index coefficient.

It is essential that the nonlinear wave equation (9) be solved as it stands without any unjustified simplifying assumptions as is typically done in nonlinear optics (see, for example, §11.3 of Ref. [8]). This is accomplished here using the split-step Fourier method introduced by Hardin and Tappert [18] by separating it into linear (dispersive and attenuating) and nonlinear parts as

\[ \nabla^2 E_{\omega_0}(\mathbf{r}, t) - \frac{n^2(\omega_0)}{c^2} \frac{\partial^2 E_{\omega_0}(\mathbf{r}, t)}{\partial t^2} = 0, \]

(10)

\[ \nabla^2 E_{NL}(\mathbf{r}, t) + \gamma|E_{NL}(\mathbf{r}, t)|^2 E_{NL}(\mathbf{r}, t) = 0, \]

(11)
respectively. The linear part (10) can then be transformed back into Eq. (7) and from there into the Helmholtz equation [cf. Eq. (2)]
\[(\nabla^2 + k^2(\omega)) \tilde{E}_D(r, \omega) = 0,\] (12)
where the subscript \(D\) indicates that this describes the linear dispersive part of the wave field. The pair of wave equations (11) and (12) then describe nonlinear dispersive pulse propagation, including diffraction effects, without the usual approximations associated with the group velocity method. An analogous formulation has been presented by Laine and Friberg [19].

Attention is now restricted to plane wave propagation in the positive \(z\)-direction through the nonlinear dispersive material. Equations (11) and (12) then admit the respective solutions
\[E_{NL}(z + h, t) = e^{i\sqrt{\gamma(E_{NL}(z))}h} E_{NL}(z, t),\] (13)
\[\tilde{E}_D(z + h, \omega) = e^{-i\tilde{\kappa}(\omega)h} \tilde{E}_D(z, \omega),\] (14)
where \(h\) is the numerical step size taken in the split-step method. The step size is taken here as \(h = z_d/5\), where \(z_d = \alpha^{-1}(\omega_0) = c/(\alpha_0 n_1(\omega_j))\) is the \(e^{-1}\) absorption depth. Notice that \(z_d \simeq 12.18m\) for the material parameters used in this paper with \(\gamma = 1\).

In each numerical simulation presented here, the pulse was propagated 5 absorption depths in order to fully display the influence of the precursor fields on the nonlinear pulse dynamics. In this numerical simulation, Eq. (14) is first applied to the pulse spectrum at \(z = z_j\) and the result is then transformed to give \(E(z_j + h, t)\). This linear propagation step is then followed by a nonlinear propagation step over the distance \(h/2\) through Eq. (13), as suggested by Agarwal [17] in order to improve accuracy of the split-step Fourier method. Notice that after the application of the nonlinear operator in Eq. (14), the real part of the resultant field is taken in order to assure physical results.

3. Heaviside step-function signal evolution

Consider first a comparison of the dynamical field evolution of a Heaviside unit-step function modulated signal \(E_H(0, t) = u_H(t) \sin(\omega_0 t)\) with \(u_H(t) = 0\) for \(t < 0\) and \(u_H(t) = 1\) for \(t > 0\) in the linear and cubic nonlinear cases, illustrated in Fig. 2 at five absorption depths \((z = 5z_d)\). The linear result is precisely that described by the modern asymptotic theory [8, 10] and, as the conditions of Sommerfeld’s relativistic causality theorem are satisfied in the linear case, \(E_H(z, t) = 0\) for all superluminal space-time points \(\theta < 1\) with \(z > 0\). The propagated wave field then arrives at the luminal space-time point \(\theta = 1\) with the onset of the Sommerfeld precursor \(E_{HS}(z, t)\), followed by the slower Brillouin precursor \(E_{HB}(z, t)\) whose peak amplitude point travels at the velocity \(v_B = c/\theta_0 = c/n(0)\), decaying only as \(z^{-1/2}\), which is then followed by the main signal \(A_{HB}(z, t)\). The nonlinear signal evolution is remarkably similar, vanishing for superluminal space-time points \(\theta < 1\), the propagated wave field arriving at \(\theta = 1\) with the onset of a Sommerfeld precursor that is essentially identical with that in the linear case. This is then followed by a Brillouin precursor whose peak amplitude is \(\sim 82\%\) of that in the linear case with a nearly identical oscillation frequency. Notice that the cubic nonlinearity generates a small amplitude frequency component at \(3\omega_0\) into the propagated wave field as the signal arrival at \(\omega = \omega_0\) is approached. The propagated signal spectrum presented in Fig. 3 shows that the cubic nonlinearity generates frequency components at the odd harmonics \(3\omega_0, 5\omega_0, 7\omega_0, \ldots\) of the carrier frequency as well as filling in the linear spectral loss in the absorption band about \(\omega_0\). This odd-harmonic frequency structure, which is missing when the quadratic wave number approximation in Eq. (6) is employed, may account for the reduced energy evident in the precursor field structure in the nonlinear case.
Fig. 2. Heaviside step-function signal response of the dispersive material at 5 absorption depths with (green curve) and without (blue curve) the cubic nonlinearity included.

Fig. 3. Propagated spectra of the Heaviside step function signal illustrated in Fig. 2 for the linear (solid blue curve) and nonlinear (solid green curve) dispersion cases.
4. Gaussian envelope pulse evolution

The second type of input signal considered here is the gaussian envelope modulated signal

\[ E_g(0,t) = \exp \left( -\frac{t^2}{\tau_0^2} \right) \sin(\omega_c t + \varphi), \tag{15} \]

where \( \varphi = 0 \) when \( N_{\text{osc}} < 1 \) (in order to ensure zero pulse area) and \( \varphi = 3\pi/2 \) when \( N_{\text{osc}} \geq 1 \) (so that the maxima of the envelope and signal coincide when an integer number \( N_{\text{osc}} \) of oscillations are contained in the pulse width). Here \( 2\tau_0 \) denotes the temporal width of the gaussian envelope at the \( e^{-1} \) amplitude points. In the numerical examples presented here, the initial pulse width \( 2\tau_0 \) was varied from \( \frac{1}{2}T_c \) to \( 100T_c \), where \( T_c \equiv 2\pi/\omega_c \) is the oscillation period of the carrier wave. For the below resonance carrier frequency \( \omega_c = 1 \times 10^{15}\text{r/s} \), this corresponds to initial pulse widths ranging from the ultrashort \( \frac{1}{2}T_c \simeq 3.14\text{fs} \) to the narrowband \( 100T_c \simeq 628\text{fs} \). In all cases considered, the computed initial pulse area is found to be less than the machine epsilon (\( \sim 2.2 \times 10^{-16} \)). The spectral magnitude \( |\tilde{u}(\omega)| \) for each extreme case is illustrated in Fig. 4 in reference to the linear material phase dispersion. The dashed curve in the figure describes the magnitude of the scaled Heaviside step function signal spectrum \( |\omega - \omega_0|^{-1} \). Notice that the ultrashort \( \frac{1}{2}T_c \) pulse spectrum (violet shaded region) is ultrawideband in comparison to the material dispersion, whereas the ultrashort \( 3T_c \) pulse spectrum (red shaded region) is wideband below resonance and the narrowband pulse spectrum (blue shaded region) is quasimonochromatic. The Sommerfeld and Brillouin precursor fields that are characteristic of the full material dispersion will clearly dominate the field evolution in the ultrawideband but not in the narrowband case. Because the intermediate wideband case fills the spectral region below the material resonance, only the low-frequency Brillouin precursor will be present in that case.

![Spectra Comparisons](image)

Fig. 4. Comparison of the spectral magnitude \( |\tilde{u}(\omega)| \) of the gaussian pulse envelope for the ultrawideband \( 2\tau_0 \simeq 3.14\text{fs} \) (magenta), the wideband \( 2\tau_0 \simeq 18.8\text{fs} \) (red), and the narrowband \( 2\tau_0 \simeq 628\text{fs} \) (blue) pulse cases. The linear material phase dispersion (green curve) and ultrawideband spectrum \( |\omega - \omega_0|^{-1} \) (dashed curve) plots are included for reference.
It is expected that the precursor fields will begin to become negligible in the total pulse evolution when the maximum slope of the initial pulse envelope function

$$u(t) = e^{(-t^2/\tau_0^2)}$$

is on the order of or greater than the medium relaxation time $T_r \sim \delta^{-1} [8]$. The first and second time derivatives of this function are given respectively by

$$u'(t) = \frac{-2t}{\tau_0^2} e^{(-t^2/\tau_0^2)}, \quad u''(t) = \frac{-2}{\tau_0^2} e^{(-t^2/\tau_0^2)} \left( \frac{2t^2}{\tau_0^2} - 1 \right).$$

The inflection point of $u'(t)$ is given by the appropriate zero of $u''(t)$, which has zeroes given by $t = \pm \tau_0/(2)^{1/2}$. Substitution of the negative root, which corresponds to the maximum, into the expression for $u'(t)$ and equating the result to $\delta$ then yields the critical value

$$\tau_c = \delta^{-1} \sqrt{2/e} \approx 1.43 \times 10^{-14} \text{s},$$

which corresponds to a minimum $N_{oc} = 4.6$ for $\omega_r = 1 \times 10^{15} \text{r/s}$. Below this critical value, either one or both of the precursor fields will be fully realized in the propagated wave field, depending on the value of the pulse carrier frequency $\omega_c$ in comparison to the medium resonance frequency $\omega_0$. On the other hand, as $\tau_0$ is increased above $\tau_c$, the observed pulse distortion will approach that described by the group velocity approximation, as is now shown.

A comparison of the computed linear and nonlinear pulse structure due to an input 3 oscillation gaussian pulse ($2\tau_0 \approx 18.8\text{fs}$) is presented in Fig. 5 at 5 absorption depths ($z = 5\tau_0$) into the dispersive medium whose linear frequency dispersion is illustrated in Fig. 1, the full dispersion response being employed in both cases. Because the pulse carrier frequency $\omega_c$ is sufficiently below the material resonance frequency $\omega_0$ so that there is negligible spectral energy above $\omega_c$, as seen in Fig. 4, the high-frequency Sommerfeld precursor is essentially absent from the propagated pulse, the pulse structure then being dominated by the Brillouin precursor [11]; as the initial pulse width $2\tau_0$ is decreased below a single oscillation, however, and the initial pulse spectrum becomes increasingly ultrawideband, the Sommerfeld precursor becomes increasingly dominant in the propagated field structure (see Fig. 6). Comparison of the propagated linear and nonlinear pulse shapes presented in Fig. 5 shows that the nonlinearity primarily decreases the amplitude of the gaussian Brillouin precursor evolution, the peak amplitude being decreased to $\sim 82\%$ of its linear value, the same amount obtained for the step function signal. This decrease may then be attributed to the generation of odd-harmonics by the nonlinearity.

The sequence of figures presented in Figs. 6–12 provides a comparison of the nonlinear gaussian pulse evolution at 5 absorption depths as the initial pulse width $2\tau_0$ is increased from the ultrawideband to the quasimonochromatic spectral extremes when the full and approximate [see Eq. (6)] dispersion relations are employed in the numerical propagation model. For the ultrawideband 3.14 fs pulse case illustrated in Fig. 6, the propagated gaussian pulse has separated into an above resonance gaussian Sommerfeld precursor and a below resonance gaussian Brillouin precursor component, as described in [11]. Notice that the group velocity approximation accurately describes just the trailing edge of the gaussian Brillouin precursor component, its peak amplitude being $\sim 10\%$ of the actual peak amplitude value.

As the initial pulse width $2\tau_0$ is increased with $\omega_c$ fixed below resonance, the Sommerfeld precursor component becomes negligible in comparison to the Brillouin precursor component (the opposite will be found when the pulse carrier frequency is situated sufficiently far above resonance), as seen in Fig. 7 for the 18.8 fs pulse case and Fig. 8 for the 31.4 fs pulse case. In both cases, the quadratic group velocity approximation only describes the trailing edge of the Brillouin precursor component with any accuracy. At this point when the critical pulse width
Fig. 5. Comparison of the propagated linear (blue curve) and nonlinear (green curve) gaussian pulses at 5 absorption depths \((z = 5z_d)\) in a single resonance Lorentz-model dielectric with below resonance carrier frequency \(\omega_c = 0.416\omega_0\).

Fig. 6. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the \(N_{osc} = 0.5\) gaussian envelope case \((2\tau_0 \simeq 3.14\text{fs})\) with \(\tau_0/\tau_c \simeq 0.11\).
Fig. 7. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 3$ gaussian envelope case ($2\tau_0 \simeq 18.8\text{fs}$) with $\tau_0/\tau_c \simeq 0.66$.

Fig. 8. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 5$ gaussian envelope case ($2\tau_0 \simeq 31.4\text{fs}$) with $\tau_0/\tau_c \simeq 1.10$. 
Fig. 9. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 10$ gaussian envelope case ($2\tau_0 \simeq 62.8 fs$) with $\tau_0/\tau_c \simeq 2.20$.

Fig. 10. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 15$ gaussian envelope case ($2\tau_0 \simeq 94.3 fs$) with $\tau_0/\tau_c \simeq 3.30$. 

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Fig. 11. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 20$ gaussian envelope case ($2\tau_0 \simeq 125.7\, fs$) with $\tau_0 / \tau_c \simeq 4.39$.

Fig. 12. Comparison of the nonlinear gaussian pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 100$ gaussian envelope case ($2\tau_0 \simeq 628.3\, fs$) with $\tau_0 / \tau_c \simeq 22.0$. 
value $\tau_c$ is exceeded, a gradual transition from the wideband, precursor dominated behavior to the narrowband quasimonochromatic behavior is observed, as evidenced in Figs. 9–12.

In Figs. 10–12 as the initial pulse spectrum becomes increasingly quasimonochromatic, the group velocity description provides a reasonable approximation to the actual pulse shape, the major two errors appearing in the underestimated pulse amplitude due to the overestimated material attenuation in the quadratic approximation and the slight phase shift appearing at the leading edge of the pulse. Finally, notice that special care had to be taken in modeling the 100 oscillation gaussian pulse case (Fig. 12) as high frequency numerical noise in the initial pulse spectrum was amplified by the cubic nonlinearity to produce a secondary pulse. This numerical error source was eliminated here by filtering the initial pulse spectrum to eliminate this noise, as illustrated in Fig. 13.

A comparison of the nonlinear space-time evolution of the envelope of the 3 oscillation gaussian pulse with initial pulse width $2\tau_0 \simeq 18.8 fs$ using the full material dispersion and the quadratic approximation of this dispersion is presented in Figs. 14 and 15, respectively, for the initial pulse evolution as $z$ increases from 0 to $z_d$, and in Figs. 16 and 17 for the mature pulse evolution [6, 8] as $z$ increases above $z_d$. The evolution of the wideband pulse into a gaussian Brillouin precursor with minimal attenuation is clearly evident in Figs. 14 and 16 (full dispersion model) while it is noticeably absent in Figs. 15 and 17 (quadratic approximation of the dielectric dispersion). In both cases there is a sharp drop in the pulse amplitude as the propagation distance increases to a single absorption depth. This is then followed by a transition to the precursor dominated field evolution in the full dispersion case illustrated in Figs. 14 and 16, where the peak amplitude decay switches from a supra-exponential to a sub-exponential decay, as illustrated by the blue curve in Fig. 18. Notice that the peak amplitude decay for the group velocity approximation, indicated by the green curve in Fig. 18, remains below the Beer’s law limit of pure exponential decay (indicated by the dashed black curve in the figure) over the propagation distance domain $z/z_d \in [0, 5]$ considered in the numerical study presented.

Fig. 13. Filtered and unfiltered spectra for the initial 100 oscillation gaussian envelope pulse.
Fig. 14. Initial nonlinear space-time evolution of the envelope of a 3 oscillation gaussian pulse with initial pulse width $2\tau_0 \simeq 18.8/\text{fs}$ using the full material dispersion. The time axis spans the interval $\Delta t = 125/\text{fs}$.

Fig. 15. Initial nonlinear space-time evolution of the envelope of a 3 oscillation gaussian pulse with initial pulse width $2\tau_0 \simeq 18.8/\text{fs}$ using the quadratic approximation of the linear material dispersion. The spurious side lobes are due to the manner in which the envelope is numerically constructed from the pulse shape.
Fig. 16. Nonlinear space-time evolution of the envelope of a 3 oscillation gaussian pulse with initial pulse width $2\tau_0 \approx 18.8\,\text{fs}$ using the full material dispersion. The time axis spans the interval $\Delta t = 125\,\text{fs}$.

Fig. 17. Nonlinear space-time evolution of the envelope of a 3 oscillation gaussian pulse with initial pulse width $2\tau_0 \approx 18.8\,\text{fs}$ using the quadratic approximation of the linear material dispersion. The time axis spans the interval $\Delta t = 125\,\text{fs}$. 
Fig. 18. Peak amplitude decay of a 3 oscillation gaussian pulse with initial pulse width $2\tau_0 \simeq 18.8 \text{fs}$ using the full material dispersion (blue curve) and using the quadratic approximation of the linear material dispersion (green curve). The black dashed line describes the Beer's law exponential decay limit $e^{-z/z_d}$ for comparison.

Here. The $z^{-1/2}$ algebraic decay that is a characteristic of the Brillouin precursor in the linear case [6, 8] is greatly exceeded in the nonlinear case, the numerically determined slope $p$ of the full dispersion curve in Fig. 18, which describes the $z^{-p}$ peak amplitude decay, decreasing from $p \sim 0.56$ at $z/z_d \sim 0.5$ to $p \sim 0.067$ at $z/z_d \sim 5$. Again, this is due to the cubic nonlinearity which also significantly affects the peak amplitude decay in the group velocity approximation as $z/z_d$ increases above unity.

5. Other pulse shapes

Similar results are obtained for other canonical pulse shapes. As an example, the propagated field structure due to an input hyperbolic secant envelope pulse

$$E_{\text{sech}}(0,t) = \text{sech}(2t/\tau_0) \cos(\omega_c t)$$

is illustrated in Fig. 19 when $\tau_0 = 3 \times 10^{-15} \text{s}$ and $\omega_c = 1 \times 10^{15} \text{r/s}$, which corresponds to the 3 oscillation gaussian pulse case illustrated in Fig. 7. As can be seen, the two results are quite similar in pulse structure, this being due to the dominance of the Brillouin precursor.

6. Conclusion

The detailed numerical results presented here have served to establish the following conclusions for the case of a causally dispersive absorptive medium with a nondispersive cubic nonlinearity:

(i) the precursor fields that are a characteristic of the linear material dispersion persist in the nonlinear case;

(ii) these precursor fields dominate gaussian pulse evolution when the initial pulse width $2\tau_0$ is on the order of or less than the critical pulse width $2\tau_c$, where $\tau_c \sim \delta^{-1}$ is the relaxation time.
Fig. 19. Comparison of the nonlinear hyperbolic secant pulse distortion using the full (blue curve) and quadratic approximation (green curve) of the linear material dispersion at 5 absorption depths for the $N_{osc} = 3$ envelope case ($2\tau_0 \simeq 18.8 fs$) with $\tau_0/\tau_c \simeq 0.66$.

of the medium response [see Eq. (18)];
(iii) as found in the linear case for the gaussian envelope pulse [11], these precursor fields result in the propagated pulse breaking up into above resonance (Sommerfeld precursor) and below resonance (Brillouin precursor) sub-pulses when the initial pulse width is sufficiently small that $2\tau_0 \ll 2\tau_c$;
(iv) the characteristic $z^{-1/2}$ peak amplitude decay of the gaussian Brillouin precursor in the linear case is enhanced by the cubic nonlinearity;
(v) the group velocity description using the quadratic dispersion approximation fails to accurately describe gaussian pulse evolution when the initial pulse width $2\tau_0$ is on the order of or less than the critical pulse width $2\tau_c$ (the accuracy does not improve with the inclusion of higher-order terms);
(vi) in addition, because the group velocity approximation overestimates the linear material attenuation away from the pulse carrier frequency $\omega_c$, the odd-order harmonics introduced by the cubic nonlinearity are absent in the group velocity description of the pulse evolution.

It is expected that the precursor fields will strongly influence the predicted pulse dispersion for other types of nonlinearity in the ultrashort/ultrawideband pulse regime. The intent of this paper is to show the central role that the precursor fields play in nonlinear pulse dynamics and thereby instigate a careful reformulation of nonlinear pulse propagation in the ultrashort pulse limit by other researchers in the nonlinear optics community.

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