Impacts of surface elevation on the growth and scaling properties of simulated river networks

Jeffrey D. Niemann a,*, Rafael L. Bras b, Daniele Veneziano b, Andrea Rinaldo c

a Department of Civil and Environmental Engineering, The Pennsylvania State University, University Park, PA 16802, USA
b Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
c Istituto di Idraulica “G. Poleni,” Università di Padova, Padova, Italy

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Abstract

We investigate the connection between surface elevation and the growth and scaling of river networks. Three planar models (Scheidegger, Eden, and invasion percolation) are first considered. These models develop aggregating networks according to stochastic rules but do not simulate erosion because the network growth is independent of the surface elevation. We show that none of these planar growth models produces scaling results consistent with observations for natural river basins. We then modify the models to include elevation, simulating the effects of fluvial erosion by enforcing the slope-area relationship. The resulting configurations have scaling properties that still depend on the model (Scheidegger, Eden, or invasion percolation) but are closer to natural river networks when compared with those from the planar growth rules. We conclude that inclusion of the vertical dimension in these three models is critical for explaining the formation and regularities of fluvial networks. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Primarily since the work of Mandelbrot (1983), scale invariant structures have been observed in many fields of the natural sciences. River networks are well-known examples of such structures, and properties such as Hack’s law (Hack, 1957) and Horton’s bifurcation ratio (Horton, 1945) are commonly cited as evidence for some form of scale invariance (Rodriguez-Iturbe and Rinaldo, 1997). One approach to understand river basin scaling is through the process by which river networks grow to fill an initially undrained region (we refer to this as the network’s mode of growth). A classical example is Scheidegger’s model (Scheidegger, 1967), which develops directed networks with particular scaling properties. Several other models have been adapted from well-known cluster growth algorithms such as Eden growth (Eden, 1961) and invasion percolation...
(Wilkinson and Willemsen, 1983). For example, Howard (1971) explored a model for river networks based on local branching properties that includes the Scheidegger and Eden schemes among its variants, whereas Stark (1991) proposed an invasion percolation model in which a river network grows by capturing the adjacent point with the lowest substrate strength.

The above models operate only on the geographical plane and rely solely on the mode of growth to determine basin structure. In real basins, the river network configuration may be affected also by the surface elevation during the process of growth. The flow directions throughout the network must be consistent with the topographic surface and therefore cannot be assigned randomly as is often done in planar models. In addition, since the expected value of channel slope at a given contributing area $E[S]$ is observed to vary with contributing area $A$ as $E[S] \propto A^{-\theta}$ for some $\theta > 0$ (Flint, 1974), there is a feedback mechanism whereby the network configuration affects the surface elevation. As a point captures more area, fluvial erosion decreases the slope thus making the point more prone to capture additional area. This dynamic can cause changes in the mode of basin growth and subsequent reorganization of the basin structure.

In this paper, we examine whether inclusion of the vertical dimension impacts the mode of network growth and the resulting basin scaling properties for three simple models—Scheidegger, Eden, and invasion percolation. First, we characterize the scaling properties of networks developed by the original planar models. In addition, since the expected value of channel slope at a given contributing area $E[S]$ is observed to vary with contributing area $A$ as $E[S] \propto A^{-\theta}$ for some $\theta > 0$ (Flint, 1974), there is a feedback mechanism whereby the network configuration affects the surface elevation. As a point captures more area, fluvial erosion decreases the slope thus making the point more prone to capture additional area. This dynamic can cause changes in the mode of basin growth and subsequent reorganization of the basin structure.

In this paper, we examine whether inclusion of the vertical dimension impacts the mode of network growth and the resulting basin scaling properties for three simple models—Scheidegger, Eden, and invasion percolation. First, we characterize the scaling properties of networks developed by the original planar models and determine whether they are adequate for river basins. We confirm and extend previous results for these models (Huber, 1991; Takayasu et al., 1991; Cieplak et al., 1998) and demonstrate the importance of examining both topological and geometrical characteristics when comparing with natural basins. Second, we investigate how the scaling properties vary when processes associated with surface elevation are included in the models. We account for elevation using erosion models similar to that of Rinaldo et al. (1993). Under certain conditions, the cluster frontiers of the new models are identical to their planar counterparts, indicating identical modes of cluster growth. However, even under such conditions, the choice of flow direction is made according to the topography and the elevation field is adjusted using the slope–area relationship, resulting in different river networks. These experiments show that the scaling exponents depend on both the mode of growth and the process of slope–area enforcement.

The organization of the paper is as follows. A background section (Section 2) describes the properties used to characterize the scaling of real and simulated basins. The planar Scheidegger, Eden, and invasion percolation models are briefly presented in Section 3. Their simulated basins are compared with typical natural basins in Section 4. Section 5 presents the models with elevation included, and Section 6 analyzes the results from these models. The final section (Section 7) summarizes the main results and states the conclusions.

2. Methods of analysis

To characterize the scaling properties of river networks, two sets of parameters are used. The first set (Tokunaga, 1978; Peckham, 1995) quantifies the topological characteristics of the network. Strahler's stream ordering is used to assign an order to each branch of the network. Branches that begin at a stream source are labeled as order one. When two streams of equal order merge, the resulting stream is given an order value that is one higher than its tributaries. When two streams of differing order meet, the resulting stream is assigned the larger of the two upstream orders (Strahler, 1957). The topology of the tree can then be described through a lower triangular matrix $T_{i,k}$ in which the elements give the average number of side tributaries of order $k$ flowing into a stream of order $i$. Topological self-similarity requires that the elements of any given diagonal be identical. Specifically, $T_{i,i} = T_{i}$ for all $i$. It has been further suggested that the $T_{i}$ values can be related as $T_{i} = bc^{i-1}$, where $b$ and $c$ are constants (Tokunaga, 1978; Peckham, 1995).

The second set of parameters includes three scaling exponents which describe the topological and geometrical scaling of river networks: $\beta$, $h$, and $H$ (Maritan et al., 1996; Rigon et al., 1996; Veneziano and Niemann, 2000). The exponent $\beta$ characterizes
the distribution of contributing area \( A \) for points regularly spaced over the basin. One can write:

\[
P[A \geq a] = a^{-\beta}f(a/a_{\text{max}})
\]  

(1)

where \( f(a/a_{\text{max}}) \) describes the deviation from log-linearity as \( a \) approaches the area of the basin \( a_{\text{max}} \) (Maritan et al., 1996). \( \beta \) is largely determined by network topology (de Vries et al., 1994). The second parameter, \( h \), is the exponent in Hack’s law (Hack, 1957), which states that \( E[L] \propto A^h \) where \( E[L] \) is the expected main stream length for a sub-basin of area \( A \). Using a finite-size scaling argument, Rigon et al. (1996) have extended the mean value relation to moments of any order \( q \):

\[
E[L^q]/E[L^{q-1}] \propto A^h 
\]  

(2)

where \( q \) is an integer exponent. The third parameter is the Hurst exponent \( H \), which relates the Euclidean width of a basin \( L_\perp \) to its Euclidean length \( L_1 \). Here \( L_\perp \) is defined as the straight line distance between the mainstream source and the basin outlet, but other definitions are possible. One can write

\[
E[L_\perp^q]/E[L_\perp^{q-1}] \propto A^{1/(H+1)} 
\]  

(3)

\( H = 1 \) is required for self-similarity of basin shapes. The three parameters, \( \beta \), \( h \), and \( H \), can be estimated by plotting the relations described by Eqs. (1)–(3) in log–log space, and performing linear regressions.

### 3. Planar models

Scheidegger (1967) proposed one of the first quantitative models for the planar development of river networks. The model operates on a hexagonal lattice and develops channels from an edge of the simulation domain that forms a line of basin outlets. At each iteration, an entire row of adjacent neighbors is simultaneously captured, and each of these points is randomly assigned a flow direction toward one of the two neighbors already within the basin. This model was originally presented as a way to develop river networks, but it may also be viewed as a simple cluster growth rule. The open boundary serves as a series of seed points, and the clusters, which correspond to the river basins, grow by a pixel layer at each iteration. Fig. 1a shows an intermediate snapshot and the final configuration for a simulation with the Scheidegger model. The model produces uniform headward growth and the aggregate cluster frontier remains smooth during development. Each channel tends to follow a relatively straight path between its source and outlet.

The second planar model used here is a well-known variant of Eden cluster growth (Eden, 1961) with the addition of a simple scheme to assign flow directions as points are added to the cluster. The model operates on a square lattice and begins with a cluster seed point. This point is located at the border and represents the outlet of the growing basin. At each iteration, a single point is randomly selected from the neighbors of the existing basin. The point is captured and given a flow direction toward a randomly selected neighbor already within the basin. Fig. 1b shows the growth of an Eden network with an outlet specified at the corner of a square domain. At all stages of growth, the basin is quite compact with growth occurring at a well defined but irregular basin frontier. Advancement of the cluster frontier is more erratic than in Scheidegger’s model, and only the very large channels are directed toward the outlet.

Both the Scheidegger and Eden models conform to Howard’s concept that river networks expand headward through an intense “wave of dissection” (Howard, 1971). In fact, if one alters Scheidegger’s model to operate on a square lattice and to allow a seed at any given location, then this Scheidegger variant and the Eden model above differ only by the number of random neighbors added at each iteration. The two models are therefore end members of Howard’s numerical growth procedure in which the number of points added to the cluster at a given iteration is controlled.

While the above models suggest that channels grow essentially by chance, invasion percolation channels grow by following paths of least resistance. This process is simulated by assigning an independent random number to each point on a square lattice at the beginning of the simulation. After the initial outlet (or outlets) has been specified, the basin grows by capturing the adjacent neighbor with the highest number. In order to define a drainage network, the added point is assigned a flow direction toward a
Fig. 1. Channel network growth using the models of (a) Scheidegger, (b) Eden, and (c) invasion percolation networks. Left panels show the cluster and network configurations after 80 iterations for Scheidegger, about 25,000 iterations for Eden, and 8000 iterations for invasion percolation. Right panels show the final spanning networks. Points outside the clusters are shaded gray, and points within the clusters are shaded according to their contributing areas. The domain size in all panels is 200 × 200 pixels.

randomly selected neighbor within the basin. Fig. 1c shows an example basin growing by invasion percolation. At intermediate stages, the basin is not compact and has a very erratic frontier. The pattern of growth differs considerably from Howard’s conceptual model because the main channels develop first,
cutting well into the simulation domain before the smaller tributaries are defined. Instead, this style of growth is consistent with Glock’s concept that fluvial networks first “elongate” to capture new territory and then “elaborate” the minor tributaries (Glock, 1931).

4. Results for the planar models

Using each of the previous models, a suitable ensemble of runs was generated for analysis. In the case of Scheidegger’s model, five runs were made on a $500 \times 2000$ pixel lattice with an open boundary along one of the shorter sides. For the other two models, five runs were made on a $500 \times 500$ pixel domain with an outlet specified at one corner. These domain sizes were selected so that power laws could be observed over at least two orders of magnitude. For clarity, the example networks shown in Fig. 1 have smaller domains ($200 \times 200$).

Fig. 2 shows results of the topological analysis for the three simulation ensembles. Each line gives the values of $T_{\omega, i, v}$, for fixed $i$ and variable $\omega$. Topological self-similarity requires that such lines be horizontal, which is observed with good approximation over a range of stream orders $\omega$. For $\omega > 6$, a significant decrease in the number of side tributaries is observed (the highest stream orders are not shown in the figure). The decrease is due to the domain shape, which narrows as one approaches the outlet. This causes many high order streams to join near the outlet, and consequently the higher order streams have short lengths and few lower order tributaries (see Fig. 1). There is also an excess of first order tributaries for streams of any order. Streams of order one are likely to depend heavily on the numerical discretization of the domain, which largely controls the number of stream sources. Over the range of orders that display self-similarity, the $T_i$ values are highest for Scheidegger’s model and lowest for invasion percolation. Assuming the model $T_i = b c^{i-1}$, one obtains $(b = 1.3, c = 3.3)$ for Scheidegger, $(b = 1.3, c = 2.9)$ for Eden, and $(b = 1.0, c = 2.7)$ for invasion percolation as best-fit values. In all three cases, only a small number of tributary orders are available for comparison, but the model fits the data.

![Figure 2](image_url)
Peckham (1995) analyzed two natural basins and found $b = 1.2$, $1.2$ and $c = 2.4$, $2.7$. These results indicate that the Scheidegger and Eden models develop basins with more side tributaries than real river networks, whereas invasion percolation networks are more similar to real basins in this respect.

One can also examine the bifurcation ratios $R_b$ (Horton, 1945). The $R_b$ values for Scheidegger, Eden, and invasion percolation are $R_b = 5.2$, $5.0$, $4.6$, respectively (Fig. 2 inset). Tarboton et al. (1989) reported bifurcation ratios for nine natural basins, with a maximum of 4.7 and an average of 4.1. Peckham (1995) found $R_b = 4.5$, $4.8$. These results confirm that Scheidegger and Eden networks have more side tributaries than natural channel networks.

Figs. 3, 4 and 5 show the distribution of contributing area for the three models. At large contributing areas, finite-size effects occur for all three models, but these effects are observed over the widest range for the Eden model. Over a range of intermediate and small areas, power laws are observed with $\beta = 0.34$, $\beta = 0.41$, and $\beta = 0.39$ for Scheidegger, Eden, and invasion percolation, respectively. These values are in good agreement with those previously reported in the literature. $\beta = 1/3$ has been analytically derived for Scheidegger (Huber, 1991; Takayasu et al., 1991) and $\beta = 0.40$, $0.38$ has been observed elsewhere for Eden and invasion percolation, respectively (Cieplak et al., 1998). The proximity of the $\beta$ values for the Eden and invasion percolation models means that $\beta$ does not capture the morphological differences between the networks that are so apparent to the eye. For natural basins, Rigon et al. (1996) have found values of $\beta$ in the range $0.40–0.46$ with an average of 0.43. Hence, the values of $\beta$ for Eden and invasion percolation are at the low end of the range observed in nature but are not unreasonable. In contrast, the exponent for Scheidegger does not agree well with real basins.

Figs. 6, 7 and 8 show Hack’s law for the simulations associated with the three models. Large deviations from log-linearity are again observed for the Eden model at large scales. The reason is that the largest tributaries are forced to converge near the outlet thereby rapidly increasing the contributing area.

Fig. 3. The distribution of contributing areas within basins developed by the Scheidegger and headward models. Three cases are shown for the headward model as described in the text. Exponents are estimated over the ranges spanned by the offset regression lines.
Fig. 4. The same as Fig. 3 with data from the Eden and random models.

Fig. 5. The same as Fig. 3 with data from the invasion percolation and ranked models.
Fig. 6. The scaling of the moment ratios of main stream length with contributing area for the Scheidegger and headward models. Three cases are shown for the headward model as described in the text. The $h$ values represent averages among the four ratios of moments. They are estimated over the ranges of data spanned by the offset regression lines. The moment ratios are also offset for clarity.

Fig. 7. The same as Fig. 6 with data from the Eden and random models.
and flattening the plots (see Fig. 1). At smaller areas, power laws are approximated with $h = 0.67$ for Scheidegger and $h = 0.61$ for both Eden and invasion percolation. For each model, the exponents calculated from various moment ratios vary only slightly. The similarity of the $h$ values for Eden and invasion percolation is not surprising given the similarity of the $\beta$ values. Maritan et al. (1996) observed that $\beta + h \approx 1$ for natural networks and derived this result under reasonable assumptions on the geometry of basins. The same value $h = 0.61$ has been previously measured by Cieplak et al. (1998) for invasion percolation.

Rigon et al. (1996) reported values of $h$ for natural basins between 0.52 and 0.60 with an average of 0.55. Here again, the Scheidegger model is in poor agreement with natural river networks. The other two values of $h$ fall at the upper edge of the range but are not unreasonable. High $h$ values mean that the length of the main stream grows more rapidly with contributing area than it does in most real basins. Such an increase in length can be due either to fractal sinuosity of the streams or elongation of the basin shapes (geometric self-affinity) (Rigon et al., 1996).

Finally, the scaling of the basin shapes for the three models is shown in Figs. 9, 10 and 11 by plotting moment ratios of $L/A$ against $A$. The plots obey power laws over two orders of magnitude with markedly different exponents for the three models. The slopes of the plots give $H = 0.50, 0.69, 1.04$ for Scheidegger, Eden, and invasion percolation, respectively. The differences in the $H$ values are substantial and indicate that geometric scaling of the sub-basin shapes is a key discriminant for these models. While the power laws considered earlier were very similar for the Eden and invasion percolation models, the $H$ values of these models differ substantially. Like Scheidegger basins, Eden basins are self-affine because they become more elongated as their areas increase, whereas invasion percolation basins come close to self-similarity ($H$ close to one).

The self-affinity of Scheidegger and Eden basins can be understood by considering their modes of growth. Scheidegger’s basins grow uniformly headward, giving large channels little opportunity to cap-
Fig. 9. The scaling of the moment ratios of Euclidean distance from the main stream source to the outlet with contributing area. Data are from the Scheidegger and headward models. Three cases are shown for the headward model as described in the text. The exponents shown are averages among the four ratios of moments and were estimated over the ranges of data spanned by the offset regression lines. The moment ratios are also offset for clarity.

Fig. 10. The same as Fig. 9 with data from the Eden and random models.
ture width from smaller neighboring basins. Eden’s model produces some deviation from uniform headward growth, causing the cluster frontier to develop self-affine irregularity (Vicsek, 1992). This irregularity gives the channels that have extended further an advantage in capturing additional side area, thus making the sub-basin shapes more self-similar. In the case of invasion percolation, the main channels grow much earlier than the smaller tributaries, and self-affinity reduces to self-similarity.

The values of $H$ noted above invalidate Scheidegger and Eden growth as models for river networks. Of the 13 basins analyzed by Rigon et al. (1996), only one has a value of $H$ below 0.88 (with a value of 0.75), and the average value is $H = 0.93$. While the value of $H$ for invasion percolation is more realistic, the simulated river courses have far too much sinuosity. The deviation between $h$ and $1/(H + 1)$ can only be due to the sinuosity of the channels. Thus, one can use $h (H + 1)$ to describe the stream sinuosity (Rigon et al., 1996). For the set of basins analyzed by Rigon et al., $h (H + 1)$ is on average 1.06 and never exceeds 1.08. For invasion percolation, it is around 1.25. This result is also confirmed visually, since invasion percolation channels are far more sinuous than natural river courses (Fig. 1c).

5. Models with elevation

The previous section has shown that the mode of cluster growth affects the scaling properties of the final network configuration. It has also shown that the three planar models considered develop networks whose scaling characteristics differ from those of real river networks. In this section, we examine the effects of including elevation in the three models. We consider a scenario in which the network grows by cutting into an initially high and undrained plateau. Erosion is assumed to reduce elevations until the local shear stresses reach a threshold value. These reductions lead to elevations that agree with the slope–area law (see below). Unlike the planar models that operate only at the cluster frontier, the extended models have erosion events that can modify elevations anywhere in the simulation domain (either inside or outside the cluster), and a particular
A critical threshold point can be eroded multiple times. The erosion algorithm is based on the work of Rinaldo et al. (1993), which provides both a criterion for determining which points are vulnerable to erosion and a rule for lowering the surface elevation at eroded points. The present model differs from the one by Rinaldo et al. because both the sequencing of the erosion events and the initial conditions are controlled to reflect the planar growth models. In addition, we do not perturb the resulting basins.

Our algorithm is as follows.

(i) Specify the initial conditions. Initial elevations are assigned to all points on the lattice (hexagonal or square). The elevations of outlet points are set to zero, and the remaining elevations are high enough that the growing basins will eventually capture the entire domain. If the initial elevations were set too low, the small slopes encountered at the cluster frontier would result in shear stresses below the stability threshold. Stability at all frontier points would halt the network growth. In the simplest case, the initial surface is flat except for the outlets. In some cases (see below), white noise is superimposed to define flow directions, but the amplitude of the noise is insufficient to induce erosion. The combined cluster frontier is defined by an elevation contour just below that of the initial flat surface. To extend the invasion percolation model, an independent random value is also assigned to each point in the domain. This value is unrelated to the surface elevation and is used to determine the sequence of erosion events.

(ii) Calculate flow directions and contributing areas. At all points, the flow direction is assumed to be the direction of steepest descent. On the square lattice, eight neighbors are used. If no neighbor is below the point under consideration, the point is labeled a “pit.” Accordingly, all points on a perfectly flat surface are considered pits. Once drainage directions are assigned, contributing areas are calculated by summing the number of grid cells upstream of each point and multiplying by the cell area. Contributing area summations are terminated at pits, where discharge is assumed to infiltrate or evaporate.

(iii) Identify unstable points. Erosion can occur only at points where the shear stress \( \tau \) is above a critical threshold \( \tau_c \). Rinaldo et al., 1993). The shear stress is evaluated as \( \tau = kA^{s}S \), where \( S \) is the gradient slope and the contributing area \( A \) is used as a surrogate for the geomorphically significant flow (Wolman and Miller, 1960; Wolman and Gerson, 1978). In all our simulations, we set \( \theta = 0.5 \), which is a reasonable value for river basins (Flint, 1974).

The constants \( \tau_c \) and \( k \) control the vertical scale of the simulated topography.

(iv) Select points for erosion. Not all of the unstable points are necessarily eroded at each iteration of the algorithm. Rinaldo et al. (1993) selected only the most unstable point for erosion (the one with the largest excess stress \( \tau - \tau_c \)), whereas we use three different criteria that mimic the planar models. To generalize the Scheidegger model, a headward algorithm is used. This algorithm begins at the outlets (and pits) and progresses headward through the current network configuration. In this fashion, all unstable points with equal distance to the outlet are eroded simultaneously, and every unstable point in the domain is eroded once before the flow directions and contributing areas are updated. To generalize the Eden model, a random sequencing algorithm is used in which a single unstable point is randomly selected for erosion. Invasion percolation is generalized using a ranked algorithm. This algorithm first determines the highest assigned random number among the unstable points of the frontier. It then erodes all unstable points in the domain having assigned numbers that are greater than or equal to that value.

(v) Erode the selected points. Erosion reduces the elevation of the selected points until they are stable with \( \tau = \tau_c \). Thus, when selected, a point is assigned a new elevation \( z = z_d + (\Delta l / k)A^{-\theta} \), where \( z_d \) is the elevation of the point immediately downstream and \( \Delta l \) is the distance separating the two points. The new elevation results in a slope that is consistent with the slope–area relationship \( S \propto A^{-\theta} \).

(vi) Iterate the process by going back to step (ii). The algorithms that we have just described assign drainage directions in a way that is different from the planar models and therefore develop different river networks. However, if the initial surface is high above the outlets and perfectly smooth, the algorithms reproduce the cluster growth rules of the planar models. The height requirement ensures that all of the cluster’s neighbors can be added to the
cluster in the present iteration. If the surface elevation is too low, neighboring points may have \( \tau \leq \tau_s \) and thus be immune from erosion. The surface is required to be smooth so that the headward algorithm grows the cluster by 1-pixel layers. The headward algorithm adds all points that currently drain into the cluster, and a smooth surface implies that only the adjacent neighbors drain into the cluster. The other variants have modes of growth that are insensitive to surface roughness.

To understand the role of these two restrictions, three cases are analyzed for the headward and random models. In Case 1, the initial surface is smooth and high enough to ensure that \( \tau > \tau_s \) at all frontier points. In Case 2, the initial surface is also high, but a small amount of white noise is added to the initial elevations. In Case 3, the initial surface has the same roughness as Case 2, but its elevation is low enough that stable points appear along the frontier throughout the growth process.

6. Results for the models with elevation

Ensembles of five simulated topographies have been generated for each case using the headward, random, and ranked models. For statistical analysis, a 500 × 500 pixel domain was used for the headward model and 200 × 200 pixel domains were used for the other two models. For visual comparison below, 200 × 200 pixel domains are used in all cases. In order to mimic their planar counterparts, the headward model uses a hexagonal grid and the random and ranked models use square grids with eight neighbors.

Fig. 12 shows individual simulations of the three headward cases. In Case 1 (Fig. 12a), the headward model has a frontier identical to the Scheidegger model (Fig. 1a), but the dynamics within the growing basins develop different-looking networks. The new basins widen more rapidly, and the larger channels have some long straight segments. The widening results from a dominance of large basins that is not observed for Scheidegger’s model. For example, consider a basin that, by chance, has widened by a small amount to increase its total area above the areas of neighboring basins. In the headward model, such a basin has lower slopes along its major streams and thus lower elevations at its stream sources. Lower source points have a greater potential to capture new points by being selected as their downstream neighbors. The straightness of the larger channels is related to the widening process and the restriction of the drainage directions. The dominant channels are straight and angled so as to capture lateral area at the maximum allowable rate. At a later stage of evolution, these dominant channels capture additional lateral area by developing small side tributaries.

In Case 2 (Fig. 12b), new features are observed. During the growth, deviation from the uniform Scheidegger frontier is introduced by the white noise in the initial surface elevations. The noise produces a random pattern of internally draining networks on the initial surface. As the cluster grows, these basins are captured and eroded, thus irregularly advancing the frontier. The final networks reflect the frontier form through an increased irregularity of the main channels.

In Case 3 (Fig. 12c), a third mode of growth is observed which is associated with a distinctive frontier. Because the initial surface has a low elevation, the shear stresses along the frontier are lower than in the previous cases. If the elevations of the boundary points in the cluster are high, then the adjacent frontier points may be stable. This stability would inhibit further growth at these points and alter the frontier form. Since most of the elevation gains in a network occur along the channels with small areas, the length of such channels tends to be restricted. This causes the frontier to maintain a roughly constant distance from the larger channels, thus giving the frontier a scalloped form.

Unlike the previous cases, the final networks in Case 3 tend to have basins that dominate far from the outlets. As the growing channels extend further from the open boundary, the elevations of the sources become higher. Because basins with smaller contributing areas have steeper trunk streams, their growth is more restricted. At some distance from the outlet, only a few basins are able to continue their growth, and in the absence of competition, they capture much lateral area.

The inclusion of elevation also affects the scaling characteristics. Fig. 3 shows the distribution of contributing areas for all three headward cases and the
Fig. 12. The growth of networks using the headward model under the three cases described in the text. The left panels show the frontier and network configurations after 80 iterations, and the right panels show the final network configurations. Points outside the clusters are shaded gray, and points within the clusters are shaded according to their contributing areas.

planar Scheidegger model. The plots for the headward cases have significantly steeper slopes in log–log than the Scheidegger model. This includes Case 1, which has a mode of growth identical to Scheidegger’s model. Case 3 has $\beta = 0.47$, which is slightly higher than the range of values observed in nature.
(Rigon et al., 1996). Cases 1 and 2 show similar
distributions to Case 3, but some significant devia-
tions from power law scaling are observed in the
central portion of the distribution where good scaling
would normally be seen for natural basins.

Hack’s law is shown in Fig. 6. The results parallel
to those for the contributing area distribution. Again,
some deviation from scaling is observed for Cases 1
and 2. Case 3 clearly obeys a power law with
\( h = 0.55 \). This value is very similar to those seen in
nature (Rigon et al., 1996) and is lower than the \( h \)
value for Scheidegger.

Fig. 9 shows the dependence of the Euclidean
basin length on contributing area. For all three cases,
the contributing area associated with a given Eu-
clidean length is much larger than for Scheidegger
basins. This difference is due to the fact that the
basins have different means to capture area laterally
as described above. For Case 3, \( H = 0.86 \) is calcu-
lated, which is much larger than \( H \) for Scheidegger’s
model. The new value is within the range observed
for natural basins but remains lower than the most
common values Rigon et al., 1996. The sinuosity
index \( h(H + 1) = 1.02 \) implies that the change in
Hack’s law from Scheidegger’s model is mainly due
to the difference in basin elongation (rather than
channel sinuosity). The sinuosity index \( h(H + 1) \) is
also consistent with observations. Overall, basins
produced by the headward model are more like
natural basins than any of the planar models previ-
ously examined.

We now turn to the random model. Fig. 13 shows
examples of these basins along with snapshots taken
during their growth. The final configurations of all
three cases are visually similar. However, because
these cases have clearly visible drainage divides (like
the headward cases), their appearance is significantly
different from that of an Eden network at small
scales (compare Figs. 1b and 13). The frontiers
observed for Cases 1 and 2 are identical to Eden,
although the drainage directions and the dynamics
within the growing networks are different. The most
interesting visual difference is the distinctive frontier
in Case 3. This frontier is scalloped like the head-
ward frontier in Case 3 and is less irregular than the
previous two random cases. This frontier form re-
results from the low elevations of the initial surface.

When any point is first added to the cluster by
erosion, it has a small contributing area and therefore
is assigned a steep slope according to the slope–area
relationship. The slope remains high until the point is
randomly selected for another erosion event, despite
any increases in contributing area as the cluster
continues to grow. This delay in the updating of
slopes causes the channels throughout the basin to
have steeper slopes than one would expect from their
current areas. When the headwater points have simi-
lar elevations to the initial surface, \( \tau \) falls below \( \tau_{st} \),
limiting the ability of the cluster to capture new
points until sufficient updating within the basin has
occurred.

Unlike the dramatic changes observed between
the Scheidegger and headward models, the random
model has scaling properties similar to those of
Eden. Fig. 4 shows the distribution of contributing
areas for the three cases. All three distributions are
similar to that for Eden. As with the headward
model, deviation from scaling is observed for Cases
1 and 2, but for Case 3, we observe \( \beta = 0.42 \). This
\( \beta \) value is larger than Eden’s and is very close to the
average of those reported in Rigon et al. (1996). Fig.
7 shows Hack’s law for basins simulated with the
random model. The exponent \( h = 0.61 \) for Case 3
is identical to that for the planar Eden model. Likewise,
in Fig. 10, one finds \( H = 0.69 \), which is identical to the
\( H \) for Eden’s model. The addition of elevation to
this model did not produce exponents substantially
closer to those of natural basins. Basins simulated
with the random model continue to be too elongated
at large scales.

Finally, we turn to the ranked model. Fig. 14
shows basins developed by the ranked model for the
cases described above. Cases 1 and 2 both have
countier frontiers that are identical to invasion percol-
ation (compare Figs. 1c and 14), but their final net-
works are substantially different. The drainage di-
vides are more visible and the large channels have
less sinuosity. During early stages of growth, the
networks develop highly sinuous trunk streams like
those of invasion percolation, but as more points
become part of the basin, smaller tributaries begin to
form. Such tributaries initially have small areas and
therefore steep slopes, but they sometimes provide
much shorter paths to the basin headwaters. This
characteristic causes their source elevations to be low enough to capture the headwater territory and thus reduces the sinuosity of the largest channels. Less reorganization occurs in Case 3 because of the low elevation of the initial surface. The channels that extend early capture very little lateral territory.
and therefore have small areas along their lengths. Because this leads to steep slopes, the sources have $\tau \leq \tau_{cr}$. These channels are thus unable to grow further until sufficient elaboration and readjustment have occurred near the outlet to increase areas and reduce source elevations. This restriction leads to a more uniform mode of growth and less reorganization during the late stages of development. Although
not observed in the example shown in Fig. 14, the
final network configuration can be quite similar to
those in Cases 1 and 2.

The scaling properties developed by the ranked
model are different from those of invasion percolation
networks. Fig. 5 shows the distribution of con-
tributing areas for the ranked model, and all three
cases adhere well to power laws at intermediate
areas. For Case 3, $\beta = 0.44$ is estimated, which is
larger than the $\beta$ value for invasion percolation and
similar to the values observed in nature. Fig. 8 shows
how the main stream length scales with contributing
area for the three cases. A value $h = 0.58$ is esti-
mated for Case 3, which is also in good agreement
with the values observed in nature. Given the visual
differences described above, one suspects that the
reduction in $h$ is due to a reduction in channel
sinuosity at large scales. Thus, the reduction in $h$ is due to a decrease in
sinuosity which is reflected in $h(H + 1) = 1.14$.
This sinuosity measure is lower than for invasion
percolation but remains above the values observed
for natural basins.

It is interesting to compare the headward, random,
and ranked models for Case 3. The models all en-
force a slope–area relationship with exponent $\theta =
0.5$, but they do so using different mechanisms.
Therefore, the models have not only different modes
of cluster growth (as described above) but also dif-
ferent reorganization dynamics within the growing
basins. We find that the planar scaling properties of
the drainage networks depend on these differences.
For example, $H = 0.86$ for the headward model,
$H = 0.69$ for the random model, and $H = 0.97$ for
the ranked model. It should be noted that the differ-
ces in boundary conditions do not affect the expo-
nents. We have simulated the headward model with a
single outlet as was done for the random and ranked
models and verified that the scaling characteristics
do not change significantly.

7. Conclusions

In the first part of the paper (Sections 3 and 4),
we analyzed the topological and geometrical scaling
characteristics of Scheidegger, Eden, and invasion
percolation networks and compared them to natural
river networks. Even though the networks generated
by the three algorithms are visually distinct, the Eden
and invasion percolation networks have similar $\beta$
and $h$ values. Small differences are observed in their
topological characteristics, but the most obvious dif-
fences are in their geometrical scaling properties.
Scheidegger and Eden basins become more elong-
ated at larger scales, whereas invasion percolation
basins are essentially self-similar. The channels of
Scheidegger and Eden are nearly straight, whereas
invasion percolation channels have a very high sinu-
osity.

The geometrical differences between the Schei-
degger, Eden, and invasion percolation networks can
be understood in terms of their modes of network
growth. Uniform headward growth, as in Scheideg-
ger’s model, develops self-affine basins and non-
fractal channels. As one allows the main channels to
grow more rapidly (i.e., shifting from Howard’s con-
ceptual model to Glock’s), the basins become
increasingly self-similar with channels that are more
sinuous. Hence, for these planar models, the mode of
network growth has an influence on the scaling
characteristics of the basin.

When compared with published statistics for natu-
ral basins, our simulations show that all the planar
models are inadequate for river networks. The low $H$
values of Scheidegger and Eden basins are not ob-
served in nature, and the sinuosity of invasion perco-
lation channels is unrealistically high.

In the second half of the paper (Sections 5 and 6),
we extended the planar models to include the effects
of surface elevation on the network properties. In-
cluding elevation in the Scheidegger and invasion
percolation models significantly affects the horizon-
tal scaling properties without necessarily changing
the mode of cluster growth. In the Scheidegger
extension, the large basins have a greater ability to
capture lateral area because of how the drainage
directions are assigned and updated as the cluster
grows. In the invasion percolation extension, signifi-
cant network reorganization occurs after the initial
development, which reduces the sinuosity of the
largest channels. Much less pronounced effects are
observed when extending Eden’s model.

The simulations with elevation also show that the
mode of growth depends on certain elevation char-
acteristics of the initial surface. For the headward
model, the mode of growth depends on the roughness of the initial surface since the pre-existing drainage divides are inherited by the growing cluster frontier. For all three models with elevation, the mode of growth is affected by the height of the initial surface because frontier points must have sufficient shear stress to be eroded. The mode of growth affects the quality of the observed power laws (especially for the headward model) with the most exact power laws occurring when the initial surface is low enough to halt the erosion at some frontier points. In such cases, the scaling exponents for the headward and ranked models are found to be different from those of the Scheidegger and invasion percolation models. The new exponent values are in better agreement with those of natural basins.

We conclude that the inclusion of elevation in standard planar models of river network formation may in some cases produce important changes in the scaling properties of such networks, making them more similar to natural river networks. The effects of elevation vary with the planar growth rule and, for example, are larger for Scheidegger than for Eden. The present results are based on simple assumptions on the sequencing of erosion events, which control the points added to the cluster, and the updating of elevations within the cluster. In reality, channel networks may grow in more complex and gradual ways and display a mixture of the features observed in our simulations.

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References


