On Fractal Properties of Arterial Trees

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The question of fractal properties of arterial trees is considered in light of data from the extensive tree structure of the right coronary artery of a human heart. Because of the highly non-uniform structure of this tree, the study focuses on the purely geometrical rather than statistical aspects of fractal properties. The large number of arterial bifurcations comprising the tree were found to have a mixed degree of asymmetry at all levels of the tree, including the depth of the tree where it has been generally supposed that they would be symmetrical. Cross-sectional area ratios of daughter to parent vessels were also found to be highly mixed at all levels, having values both above and below 1.0, rather than consistently above as has been generally supposed in the past. Calculated values of the power law index which describes the theoretical relation between the diameters of the three vessel segments at an arterial bifurcation were found to range far beyond the two values associated with the cube and square laws, and not clearly favoring one or the other. On the whole the tree structure was found to have what we have termed "pseudo-fractal" properties, in the sense that vessels of different calibers displayed the same branching pattern but with a range of values of the branching parameters. The results suggest that a higher degree of fractal character, one in which the branching parameters are constant throughout the tree structure, is unlikely to be attained in non-uniform vascular structures.

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Introduction

The branching pattern of vascular systems has been under study for many years, with the aim first of mapping out that pattern and then of uncovering the principles on which it is based. So far the two tasks have been addressed somewhat separately, and it is reasonable to say that a comprehensive theory or a set of data which is able to put the two together has not emerged yet.

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A structure which has been studied widely is that of an open "tree" in which typically a single trunk divides into branches, then each of the branches in turn divides into branches and so on. This branching structure is important because of its application to botanical or arterial trees and to the ramifications of rivers. In this paper we shall be concerned with the application to arterial trees only, where it has been found further that the number of branches at each division is almost invariably two (Mall, 1906; Green, 1950; Zamir, 1976). This repeated bifurcation process gives the tree its first fractal character but only in the most rudimentary
sense. More detailed scrutiny of the hierarchic structure of the tree is required if the fractal pattern is to be examined more closely.

Early studies described the hierarchic structure of an arterial tree in terms of different categories of "branches" within it (Mall, 1906; Green, 1950), these being defined according to certain rules as in the Strahler or Weibel methods (Horton, 1945; Strahler, 1952; Weibel, 1963). Branches of the same category are then grouped together to form different "levels" of the tree. A central feature of this description is the notion of a branch as a "whole vessel" which has a well defined identity from beginning to end. This notion has led to difficulties, first because the way in which these whole vessels can be defined is not unique, and second because such vessels cannot be easily identified and measured in real tree structures (Zamir & Phipps, 1988).

Because of these difficulties more recent work has been based on the more readily accessible notion of "vessel segments", a vessel segment being defined simply as that part of a vessel between two consecutive junctions (Zamir & Phipps, 1988; Zamir & Chee, 1987). Vessel segments are uniquely defined and can be easily identified and measured in a real tree structure and, conversely, the tree structure can be mapped out more accurately in terms of these segments than in terms of whole vessels. In this mapping, a tree structure begins with a single vessel segment which divides into two segments, then each of these in turn divides into two segments and so on (Figs 1 and 2). In contrast with the classical notion of whole vessels which must be traced from beginning to end, here the identity of each vessel segment terminates as that segment divides. A vessel segment is "borne" at a vascular junction and "dies" at the next downstream junction as it gives rise to two daughter segments. The three segments form an "arterial bifurcation", which is the ubiquitous building block, the structural kernel, of an arterial tree (Fig. 2). The properties of arterial bifurcations are central to the study of arterial tree structures, and both theory and experiment have targeted these properties.

What is meant by the "levels" or "generations" of an arterial tree depends on whether the tree structure is being described in terms of the classical notion of whole vessels or in terms of vessel segments. Consequently, what is meant by "fractal properties" of an arterial tree depends fundamentally on whether the tree structure is being described by the first or the second of these methods. The purpose of the present paper is to clarify these issues and to use a set of measurements of bifurcation properties taken from the extensive tree structure of a right coronary artery to show the type of fractal properties observed in this structure.

Because of the highly non-uniform structure of this tree, the study focuses on the purely geometrical rather than statistical aspects of fractal properties. Statistically based studies have been applied successfully to the more uniform
The course of the vessel passes through an artifactally determined property of an arterial wall. Another important property of an arterial wall is the power law index $k$ which depends on the ratio of the vessel's outer to inner diameter $d$. The relationship between the power law index and the ratio of the vessel's outer to inner diameter is given by

$$\frac{d}{h} = f$$

where $h$ is the height of the vessel and $f$ is a constant that depends on the arterial wall. The power law index $k$ is given by

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governs the fluid dynamic efficiency of the bifurcation and provides a relation between the three vessel diameters of the form

\[ d_3 = d_1^i + d_2^i \]  

(3)

This relation is based on the requirement that the flow rate \( q \) through an arterial bifurcation satisfies the conservation law \( q_0 = q_1 + q_2 \), and on the assumption that a power-law relation of the form \( q \propto d^k \) exists between the flow rate in a vessel segment and the diameter of that segment. The value of \( k \) depends on the optimality criterion used to obtain the power law relation between diameter and flow. An important value, associated with the so-called "cube law" is \( k = 3 \), which corresponds to a condition of minimum rate of energy expenditure for flow and metabolic purposes, as well as invariant shear stress at different levels of the tree. Another value, associated with the "square law", is \( k = 2 \), which corresponds to a condition of zero expansion in cross-sectional area available to the flow as it progresses through the bifurcation and, hence, as it progresses from one level of the tree structure to the next.

Physiological data have so far failed to discriminate clearly between the two values, but there are some indications that the square law may prevail in some parts of the vascular system and the cube law in others (Sherman, 1981; Roy & Wildenberg, 1982; Wildenberg & Horsfield, 1983; Zamir et al., 1992). Given a value of \( k \), the expression for \( \beta \) in eqn (2) can be put in the form

\[ \beta = \frac{1 + \alpha^2}{(1 + \alpha^3)^{2\alpha}} \]  

(4)

In this form it is seen more clearly how \( \beta \) and \( k \) are related to each other, indeed as they are both measures of expansion or contraction of the cross-sectional area at a bifurcation. In particular, it is seen that \( k = 2 \) corresponds to \( \beta = 1 \), while \( k > 2 \) corresponds to \( \beta > 1 \) and \( k < 2 \) corresponds to \( \beta < 1 \).

Criteria for Fractal Properties in a Tree Structure

Fractal properties can be ascribed to an arterial tree according to different criteria and on a scale of different degrees of fractal behavior. We emphasize that, as discussed earlier, the focus in this paper is on geometrical rather than statistical aspects of fractal properties. The following two criteria provide a starting point:

(I) a vascular tree exhibits fractal properties if any sub-tree from it has precisely the same tree structure as the whole;

(II) a tree exhibits fractal properties if progression from one level of the tree to the next is based on the same branching pattern.

While they are fairly intuitive, both criteria lead to difficulties when applied to a real vascular structure and they point to the issues which must be addressed to produce more rigorous criteria.

The difficulty with the first criterion is that it does not identify clearly the structural properties of the tree which must be duplicated in the sub-tree. At one extreme it may mean no more than that the tree and sub-tree would "look" the same. At the other extreme it may require that some precise scaling can be used to render the tree and sub-tree identical in topological structure and in the location and size of branches, though clearly not in the number of branches if the two trees are finite. While the elementary form of the criterion is used widely in ascribing fractal properties to clouds, shorelines, and some biological structures (Mandelbrot, 1977; West et al., 1997; Peitgen et al., 1992), a more stringent form of the criterion is clearly required in the case of an arterial tree where function rather than anatomical appearance is the object of study (Zamir, 1988a).

The difficulty with the second criterion is that it depends critically on what is meant by a "level" of the tree. As mentioned earlier, this term has been used rather loosely in classical studies to mean "a group of vessels having the same branching order within the tree", "order" being defined by the Strahler or Weibel methods (Horton, 1945; Strahler, 1952; Weibel, 1963). When branching levels are defined in this way, however, progression from one level of the tree to the next becomes somewhat intractable. While the rules of the Shrahler and Weibel methods assign accurately the order of each vessel within a tree, the methods do not prescribe a pre-determined algorithm by which vessels of one order are connected to vessels of the next
entirely of arterial bifurcations. Measurements of the bifurcation index \( \alpha \) at 2265 bifurcations are shown in Fig. 3 where the value of \( \alpha \) at each bifurcation is plotted against the diameter of the parent vessel segment at that bifurcation. Against the full range of diameters it is seen that only at the higher end of the tree there is some bias towards smaller values of \( \alpha \), consistent with the peculiar branching pattern of the main distributing vessels of the tree. The number of vessel segments involved in this range is seen to be comparatively small since the number of distributing vessels within the tree is comparatively small. In the middle range of diameters where the bulk of vessel segments lie it is seen that values of the bifurcation index are distributed almost uniformly between 0 and 1.0, which indicates that bifurcations of uniformly mixed degrees of asymmetry persist down to vessels of very small diameters. This result is surprising since it has been generally supposed in the past that at the “depth” of an arterial tree, away from the main distributing vessels, bifurcations would be mostly symmetrical. There is no indication of such a trend in the results, in fact only a faint tendency can be observed at the limit of the lower diameter range.

Results for the area ratio \( \beta \) are shown in Fig. 4. The ratio is a measure of expansion (\( \beta > 1.0 \)) or contraction (\( \beta < 1.0 \)) in cross-sectional area available to the flow at individual bifurcations and in the tree as a whole. Again, it is seen that for the relatively small number of bifurcations in the upper diameter range there is some bias towards values of \( \beta \) near or just above 1.0, but for the bulk of bifurcations in the middle range the values of \( \beta \) assume an increasingly wide range, both higher and lower than 1.0, though with some tendency towards the higher values. For the arterial tree as a whole it has been estimated that a total expansion of 1000 occurs from the cross-sectional area of the aorta to that of the capillary bed, which requires values of \( \beta \) consistently above 1.0 at many arterial bifurcations (Zamir, 1988a). The present results indicate that much of this expansion occurs within the middle and lower diameter ranges, as is generally assumed, but that it does not occur consistently in these regions. Instead, expansion at many bifurcations is accompanied by contraction at many others.

Coupled with the value of \( \beta \) is the value of the power-law index \( k \), as defined by eqn (3). As
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Fig. 4. Area ratio $\beta$ of bifurcations along the tree structure of the right coronary artery, plotted against the diameter of the parent vessel segment at each bifurcation. The full range of the data is shown in the upper panel, with smaller ranges in the middle and lower panels.

shown earlier, a value of $k = 2.0$ (square law) represents zero change in cross-sectional area at a bifurcation, while values of $k > 2.0$ and $k < 2.0$ represent expansion and contraction in cross-sectional area, respectively. Values of $k$ based on diameter measurements in the three diameter ranges, and calculated from eqn (3), are shown in Fig. 5. In the upper diameter range values of $k$ are seen to be mainly in the 1.0–3.0 range, but increasingly higher values of $k$ occur in the middle and lower diameter ranges.

Variation of the length of vessel segments in an arterial tree is not clearly understood at present and has not been supported by sufficient data so far. Measurements of segment length are shown in Fig. 6 where they are plotted against segment diameter. There have been no theoretical grounds so far on which a direct correlation between segment length and segment diameter can be expected, and our results do not indicate any such correlation. The results show a clear limiting line of maximum length, however, for vessel segments of a given diameter, suggesting a maximum length to diameter ratio of approximately 35. A ratio of 10 is commonly used for theoretical tree structures, and a line corresponding to this value is shown for comparison. The data suggest that on average a ratio of 10 is a reasonable approximation.

Discussion and Conclusions

In the tree structure of the right coronary artery examined in this study, bifurcations were found to have a fairly mixed degree of asymmetry at all levels of the tree, including the “depth” of the tree where it has been generally supposed that they would be symmetrical. Area ratios were found to be highly mixed, having values both above and below 1.0, rather than consistently above as has been generally supposed in the past. Values of the power-law index calculated from the measured data were found to range far beyond the two values associated with the cube and square laws, and not clearly favoring one or the other. No correlation between length and diameter of vessel segments was found, but a plot of the measured data suggests that the ratio of length to diameter has a limiting maximum value of about 35, and that the average value of 10 used in approximate calculations is reasonable.
In the context of Criterion III for fractal properties, results in Figs 3–5 for α, β, k indicate clearly that these branching parameters do not

![Graphs showing data for Figs 3-5](image)

FIG. 5. Power law index k calculated for bifurcations along the tree structure of the right coronary artery and plotted against the diameter of the parent vessel segment at each bifurcation. The full range of the data is shown in the upper panel, with smaller ranges in the middle and lower panels.

While there is no indication of any correlation between length and diameter, there is evidence of an upper limit for the length as indicated by (—). The slope of the line is 35. A line (——) with a slope of 10 is shown for comparison, which corresponds to an approximation commonly used in theoretical computations, that the length of a vessel segment is 10 times its diameter.
have constant values as we progress from one diameter group to the next along the tree structure, but in the middle and lower diameter ranges they appear to have the same range of values. Thus in the middle and lower diameter ranges while there is considerable variation in the values of $\alpha$, $\beta$, $k$, the same variation occurs at all diameter levels. We conclude therefore that while the tree does not have strict fractal properties in the sense of having constant values of $\alpha$, $\beta$, $k$, it has what we might call pseudo-fractal properties in the sense that vessels of different calibers have the same range of values of these branching parameters.

Variability in the values of branching parameters is fairly common in vascular trees in the cardiovascular system and it is a major source of non-uniformity in their tree structure. Another major source of non-uniformity of many vascular trees is “incompleteness” of their tree structure in the sense that many segment positions on the map of a complete tree structure such as that in Figs 1 and 2 would not be occupied by vessel segments. Vascular trees are generally highly “skewed” in the direction of their main distributing vessels as is the case in the vasculature of the heart from which our data are obtained (Zamir, 1996). In the presence of such non-uniformity it would seem that a higher degree of fractal character, in the sense of Criterion III, may not be attainable in many trees in the cardiovascular system. The difficulties associated with Criteria I and II would be compounded by non-uniformity of the tree structure, as discussed earlier.

A distinction must be made between a vascular tree which serves a purely metabolic function such as that of the heart or the brain, and one which serves some “processing” function such as that of the lung or kidney. Because of the major difference in their functions, the two types of trees differ considerably in the degree of their uniformity and consequently in the degree of their fractal properties. Appropriate methods for studying these properties may also be different in the two cases since function must ultimately play a role in the definition of these properties. Methods based on statistical or space-filling fractals may be more appropriate for trees with a processing function, while methods based on branching geometry may be more appropriate for trees with a metabolic function.

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REFERENCES


