WATER RESOURCES RESEARCH, VOL. 27, NO. 7, PAGES 1671–1684, JULY 1991

(B)

[willgoose: channel 1]

A Coupled Channel Network Growth and Hillslope Evolution Model
1. Theory

GARRY WILLGOOSE,[1] RAFAEL L. BRAS, AND IGNACIO RODRIGUEZ-ITURBE[2]

Ralph M. Parsons Laboratory, Massachusetts Institute of Technology, Cambridge

This paper presents a model of the long-term evolution of catchments, the growth of their drainage networks, and the changes in elevations within both the channels and the hillslopes. Elevation changes are determined from continuity equations for flow and sediment transport, with sediment transport being related to discharge and slope. The central feature of the model is that it explicitly differentiates between the sediment transport behavior of the channels and the hillslopes on the basis of observed physics, and the channel network extension results solely from physically based flow interactions on the hillslopes. The difference in behavior of channels and hillslopes is one of the most important properties of a catchment. The flow and sediment transport continuity equations in the channel and the hillslope are coupled and account for the long-term interactions of the elevations in the hillslope and in the channels. Sediment transport can be due to fluvial processes, creep, and rockslides. Tectonic uplift may increase overall catchment elevations. The dynamics of channel head advance, and thus network growth, are modeled using a physically based mechanism for channel initiation and growth where a channel head advances when a channel initiation function, nonlinearly dependent on discharge and slope, exceeds a threshold. This threshold controls the drainage density of the basin. A computer implementation of the model is introduced, some simple simulations presented, and the numerics of the solution technique described.

INTRODUCTION

The flood response of a catchment to rainfall is dependent on the geomorphological form of the catchment. But the catchment runoff not only responds to catchment form, it also shapes it through the erosional processes that act during runoff events. Over geologic time the catchment form, shaped by the range of erosion events, reflects the runoff processes that occur within it. The channel network form and extent reflect the characteristics of both the hillslope and channel processes. Hydrologists have long parameterized the influence of the geomorphology on flood response [e.g., *Rodriguez-Iturbe and Valdes*, 1979]. Many geomorphologists have statistically or probabilistically described the landscape, ignoring the historic processes that created it [*Strahler*, 1964; *Shreve*, 1966]. There have been some notable exceptions [*Gilbert*, 1909; *Horton*, 1945]. The difficulty of the problem is such that the number of researchers that have attempted to unify the geomorphology and the hydrology is small [*Kirkby*, 1971; *Huggett*, 1988; *Dunne*, 1989], even though the importance of both specializations has long been recognized by geomorphologists [*Davis*, 1909, p. 268]:

> to look upon the landscape . . . without any recognition of the labor expended in producing it, or of the extraordinary adjustments of streams to structures and of waste to weather, is like visiting Rome in the ignorant belief that the Romans of today have no ancestors.

The main stumbling blocks to the fulfillment of the promise of this scientific paradigm have been the range of temporal scales (geologic versus flood event time scales) and spatial scales (catchment and channel length scales) that are important in the problem, the heterogeneity in both space and time of the dominant processes, and the problem of the unification and observation of the processes acting at these disparate scales. Physically based computer models of catchment development [e.g., *Ahnert*, 1976; *Kirkby*, 1987] are important tools in the understanding of the interactions between hydrologic process and response, primarily because of their ability to explore the sensitivity of the system to changes in the physical conditions, without many of the difficulties implicit in field studies.

The goal here is to develop a quantitative understanding of how channel networks and hillslopes evolve with time using a computer model of landscape evolution. Catchment form and hydrologic response are seen in the context of the complete history of erosional development of the catchment.

A large-scale model of catchment evolution involving channel network growth and elevation evolution has been developed. It brings together a model of erosion processes that has been theoretically and experimentally verified at small scales, with a physically based conceptualization of the channel growth process. It will be shown that neither the properties of the channel network nor the properties of the hillslopes can be viewed in isolation but must be viewed as components of a complicated large-scale nonlinear system: the drainage basin. The basic tenet of this work is that it is necessary to understand the physics of the catchment processes to be able to fully understand the catchment form and that it is necessary to ". . . identify linked process equations and so define geomorphological systems in such a way that an analytical, predictive approach can be used. . ." [*Huggett*, 1988, p. 48]. It is not claimed, nor is it intended, that the proposed model account for all the processes occurring in the catchment. Rather, a general model framework is presented which is both physically realistic and incorporates the dominant physical processes yet provides a useful tool by which the important interactions within the catchment can be examined.

[1]Now at the Department of Civil Engineering and Surveying, University of Newcastle, Newcastle, New South Wales, Australia.
[2]Now at Instituto Internacional de Estudios Avanzados, Caracas.

Paper number 91WR00935.
0043-1397/91/91WR-00935$05.00

WATER RESOURCES RESEARCH, VOL. 27, NO. 7, PAGES 1671–1684, JULY 1991

B

[willgoose:channel1]

A Coupled Channel Network Growth and Hillslope Evolution Model
1. Theory

GARRY WILLGOOSE,[1] RAFAEL L. BRAS, AND IGNACIO RODRIGUEZ-ITURBE[2]

Ralph M. Parsons Laboratory, Massachusetts Institute of Technology, Cambridge

This paper presents a model of the long-term evolution of catchments, the growth of their drainage networks, and the changes in elevations within both the channels and the hillslopes. Elevation changes are determined from continuity equations for flow and sediment transport, with sediment transport being related to discharge and slope. The central feature of the model is that it explicitly differentiates between the sediment transport behavior of the channels and the hillslopes on the basis of observed physics, and the channel network extension results solely from physically based flow interactions on the hillslopes. The difference in behavior of channels and hillslopes is one of the most important properties of a catchment. The flow and sediment transport continuity equations in the channel and the hillslope are coupled and account for the long-term interactions of the elevations in the hillslope and in the channels. Sediment transport can be due to fluvial processes, creep, and rockslides. Tectonic uplift may increase overall catchment elevations. The dynamics of channel head advance, and thus network growth, are modeled using a physically based mechanism for channel initiation and growth where a channel head advances when a channel initiation function, nonlinearly dependent on discharge and slope, exceeds a threshold. This threshold controls the drainage density of the basin. A computer implementation of the model is introduced, some simple simulations presented, and the numerics of the solution technique described.

INTRODUCTION

The flood response of a catchment to rainfall is dependent on the geomorphological form of the catchment. But the catchment runoff not only responds to catchment form, it also shapes it through the erosional processes that act during runoff events. Over geologic time the catchment form, shaped by the range of erosion events, reflects the runoff processes that occur within it. The channel network form and extent reflect the characteristics of both the hillslope and channel processes. Hydrologists have long parameterized the influence of the geomorphology on flood response [e.g., *Rodriguez-Iturbe and Valdes*, 1979]. Many geomorphologists have statistically or probabilistically described the landscape, ignoring the historic processes that created it [*Strahler*, 1964; *Shreve*, 1966]. There have been some notable exceptions [*Gilbert*, 1909; *Horton*, 1945]. The difficulty of the problem is such that the number of researchers that have attempted to unify the geomorphology and the hydrology is small [*Kirkby*, 1971; *Huggett*, 1988; *Dunne*, 1989], even though the importance of both specializations has long been recognized by geomorphologists [*Davis*, 1909, p. 268]:

> to look upon the landscape . . . without any recognition of the labor expended in producing it, or of the extraordinary adjustments of streams to structures and of waste to weather, is like visiting Rome in the ignorant belief that the Romans of today have no ancestors.

The main stumbling blocks to the fulfillment of the promise of this scientific paradigm have been the range of temporal scales (geologic versus flood event time scales) and spatial scales (catchment and channel length scales) that are important in the problem, the heterogeneity in both space and time of the dominant processes, and the problem of the unification and observation of the processes acting at these disparate scales. Physically based computer models of catchment development [e.g., *Ahnert*, 1976; *Kirkby*, 1987] are important tools in the understanding of the interactions between hydrologic process and response, primarily because of their ability to explore the sensitivity of the system to changes in the physical conditions, without many of the difficulties implicit in field studies.

The goal here is to develop a quantitative understanding of how channel networks and hillslopes evolve with time using a computer model of landscape evolution. Catchment form and hydrologic response are seen in the context of the complete history of erosional development of the catchment.

A large-scale model of catchment evolution involving channel network growth and elevation evolution has been developed. It brings together a model of erosion processes that has been theoretically and experimentally verified at small scales, with a physically based conceptualization of the channel growth process. It will be shown that neither the properties of the channel network nor the properties of the hillslopes can be viewed in isolation but must be viewed as components of a complicated large-scale nonlinear system: the drainage basin. The basic tenet of this work is that it is necessary to understand the physics of the catchment processes to be able to fully understand the catchment form and that it is necessary to ". . . identify linked process equations and so define geomorphological systems in such a way that an analytical, predictive approach can be used. . ." [*Huggett*, 1988, p. 48]. It is not claimed, nor is it intended, that the proposed model account for all the processes occurring in the catchment. Rather, a general model framework is presented which is both physically realistic and incorporates the dominant physical processes yet provides a useful tool by which the important interactions within the catchment can be examined.

[1]Now at the Department of Civil Engineering and Surveying, University of Newcastle, Newcastle, New South Wales, Australia.
[2]Now at Instituto Internacional de Estudios Avanzados, Caracas.

Paper number 91WR00935.
0043-1397/91/91WR-00935$05.00

WATER RESOURCES RESEARCH, VOL. 27, NO. 7, PAGES 1671–1684, JULY 1991

[willgoose: channel 1]

B

A Coupled Channel Network Growth and Hillslope Evolution Model
1. Theory

GARRY WILLGOOSE,[1] RAFAEL L. BRAS, AND IGNACIO RODRIGUEZ-ITURBE[2]

Ralph M. Parsons Laboratory, Massachusetts Institute of Technology, Cambridge

This paper presents a model of the long-term evolution of catchments, the growth of their drainage networks, and the changes in elevations within both the channels and the hillslopes. Elevation changes are determined from continuity equations for flow and sediment transport, with sediment transport being related to discharge and slope. The central feature of the model is that it explicitly differentiates between the sediment transport behavior of the channels and the hillslopes on the basis of observed physics, and the channel network extension results solely from physically based flow interactions on the hillslopes. The difference in behavior of channels and hillslopes is one of the most important properties of a catchment. The flow and sediment transport continuity equations in the channel and the hillslope are coupled and account for the long-term interactions of the elevations in the hillslope and in the channels. Sediment transport can be due to fluvial processes, creep, and rockslides. Tectonic uplift may increase overall catchment elevations. The dynamics of channel head advance, and thus network growth, are modeled using a physically based mechanism for channel initiation and growth where a channel head advances when a channel initiation function, nonlinearly dependent on discharge and slope, exceeds a threshold. This threshold controls the drainage density of the basin. A computer implementation of the model is introduced, some simple simulations presented, and the numerics of the solution technique described.

INTRODUCTION

The flood response of a catchment to rainfall is dependent on the geomorphological form of the catchment. But the catchment runoff not only responds to catchment form, it also shapes it through the erosional processes that act during runoff events. Over geologic time the catchment form, shaped by the range of erosion events, reflects the runoff processes that occur within it. The channel network form and extent reflect the characteristics of both the hillslope and channel processes. Hydrologists have long parameterized the influence of the geomorphology on flood response [e.g., *Rodriguez-Iturbe and Valdes*, 1979]. Many geomorphologists have statistically or probabilistically described the landscape, ignoring the historic processes that created it [*Strahler*, 1964; *Shreve*, 1966]. There have been some notable exceptions [*Gilbert*, 1909; *Horton*, 1945]. The difficulty of the problem is such that the number of researchers that have attempted to unify the geomorphology and the hydrology is small [*Kirkby*, 1971; *Huggett*, 1988; *Dunne*, 1989], even though the importance of both specializations has long been recognized by geomorphologists [*Davis*, 1909, p. 268]:

> to look upon the landscape . . . without any recognition of the labor expended in producing it, or of the extraordinary adjustments of streams to structures and of waste to weather, is like visiting Rome in the ignorant belief that the Romans of today have no ancestors.

The main stumbling blocks to the fulfillment of the promise of this scientific paradigm have been the range of temporal scales (geologic versus flood event time scales) and spatial scales (catchment and channel length scales) that are important in the problem, the heterogeneity in both space and time of the dominant processes, and the problem of the unification and observation of the processes acting at these disparate scales. Physically based computer models of catchment development [e.g., *Ahnert*, 1976; *Kirkby*, 1987] are important tools in the understanding of the interactions between hydrologic process and response, primarily because of their ability to explore the sensitivity of the system to changes in the physical conditions, without many of the difficulties implicit in field studies.

The goal here is to develop a quantitative understanding of how channel networks and hillslopes evolve with time using a computer model of landscape evolution. Catchment form and hydrologic response are seen in the context of the complete history of erosional development of the catchment.

A large-scale model of catchment evolution involving channel network growth and elevation evolution has been developed. It brings together a model of erosion processes that has been theoretically and experimentally verified at small scales, with a physically based conceptualization of the channel growth process. It will be shown that neither the properties of the channel network nor the properties of the hillslopes can be viewed in isolation but must be viewed as components of a complicated large-scale nonlinear system: the drainage basin. The basic tenet of this work is that it is necessary to understand the physics of the catchment processes to be able to fully understand the catchment form and that it is necessary to ". . . identify linked process equations and so define geomorphological systems in such a way that an analytical, predictive approach can be used. . ." [*Huggett*, 1988, p. 48]. It is not claimed, nor is it intended, that the proposed model account for all the processes occurring in the catchment. Rather, a general model framework is presented which is both physically realistic and incorporates the dominant physical processes yet provides a useful tool by which the important interactions within the catchment can be examined.

[1]Now at the Department of Civil Engineering and Surveying, University of Newcastle, Newcastle, New South Wales, Australia.
[2]Now at Instituto Internacional de Estudios Avanzados, Caracas.

Paper number 91WR00935.
0043-1397/91/91WR-00935$05.00

This paper will concentrate on the development of the model, justification of the governing equations, and illustrating the operation of the model with a sample simulated catchment. An accompanying paper [*Willgoose et al.*, this issue (*a*)] will present a nondimensionalization of this model, identify governing nondimensional numbers, and study the properties of simulated catchments.

THE PHYSICAL MODEL

The governing equations of this work are used to simulate the growth and evolution of the channel networks and the contributing hillslopes. Two variables are solved for in the plane: the elevation and an indicator variable that identifies where channels exist in space. A drainage direction is assigned to each node in the discretized space on the basis of the direction of steepest slope from node to node. These drainage directions are used to determine the area contributing to (i.e., flowing through) each node. From these areas, and thus discharge, and the steepest slopes at the nodes, continuity equations for flow and sediment transport are written. These areas and steepest slopes are also used to evaluate the channel initiation function (which may, for example, be overland flow velocity) which is then used in the channelization function to determine regions of active channel network extension. Details of the numerical implementation of the solution technique for these equations can be found in Appendix A.

The governing differential equations for elevation and channel indicator functions are

$$\frac{\partial z}{\partial t} = c_0(x, y) + \frac{1}{\rho_s(1-n)}\left(\frac{\partial q_{sx}}{\partial x} + \frac{\partial q_{sy}}{\partial y}\right) + D_z\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \quad (1a)$$

$$\frac{\partial Y}{\partial t} = d_t\left[0.0025\frac{a}{a_t} + \left(-0.1Y + \frac{Y^2}{1+9Y^2}\right)\right] \quad (1b)$$

and the constitutive equations are

$$a = \beta_5 q^{m_5} S^{n_5} \quad (2a)$$

$$q_s = f(Y) q^{m_1} S^{n_1} \quad (2b)$$

$$\begin{aligned} f(Y) &= \beta_1 O_t \qquad Y = 0 \text{ (hillslope)} \\ f(Y) &= \beta_1 \qquad Y = 1 \text{ (channel)} \end{aligned} \quad (2c)$$

and for channels,

$$Q_c = \beta_3 A^{m_3} \quad (2d)$$

$$q = Q_c/w \quad (2e)$$

$$w = \beta_4 Q_c^{m_4} \quad (2f)$$

where the equations are solved on some spatial domain Ω with boundary conditions

$$\partial z/\partial p = 0 \quad (2g)$$

on the boundary of Ω.

The variables in (1) and (2) are

t time;
x, y horizontal directions;
z elevation;
Y indicator variable for channelization ($Y = 0$, hillslope node; $Y = 1$, channel node);
a channel initiation function;
q_{sx}, q_{sy} sediment flux per unit width in the x and y directions;
Q_c discharge, in the channel;
q discharge per unit width;
w width of channels, variable with discharge;
β_1 rate constant for sediment transport;
β_4, m_4 constants relating channel width to discharge [*Henderson*, 1966];
O_t rate constant relating the sediment transport flux in the channel to that on the hillslope;
A contributing area;
$c_0(x, y)$ rate of tectonic uplift;
ρ_s density of eroded material;
n porosity of material before erosion and after deposition;
S slope in steepest downhill direction;
D_z diffusivity constant in certain transport processes;
m_1, n_1 powers of q and S in the sediment transport equation;
d_t rate constant for channel growth;
a_t channel initiation threshold;
β_5 multiplicative constant on channel initiation function;
m_5, n_5 powers on q and S, respectively, in the channel initiation function;
β_3, m_3 multiplicative constant and power, respectively, relating the characteristic discharge to the contributing area;
p direction perpendicular to the boundary of Ω.

The governing equations are nonlinear partial differential equations of two states; these two states are elevation z and an indicator variable for channelization, Y. The most important qualitative characteristic of a catchment, the branched network of channels that form the backbone of the drainage system of a basin, is thus explicitly modeled. There are five important variables distributed in space that are derived from these two states. They are the steepest downhill slope, the contributing area, the discharge, the distribution of channel initiation function, and sediment transport in space. The channel initiation function and sediment transport feed back into, as inputs, the two state equations for elevation and channelization. Thus there is a nonlinear interaction between the elevation and channelization and the channel initiation function and sediment transport in space. This interaction is the central feature of the model that drives the network growth.

The differential equation for elevation (1*a*) is a continuity equation in space for sediment transport. The first term in (1*a*) is the rate of tectonic uplift (positive upward). This term may be quite general with variability both in space and time. For instance, a spatially uniform uplift event, such as that resulting from an earthquake [*Morisawa*, 1964] can be described by

$$c_0(\mathbf{x}, t) = \bar{c}_0 \delta(t - t_0) \quad (3)$$

where $\bar{c}_0$ is the uplift resulting from the tectonic event, t_0 is the time at which the event occurred, and $\delta(t)$ is the dirac delta function.

Likewise, it could be an uplift that occurs continuously with time but is variable in space, such as that resulting from continuous bulging of the continental crest [e.g., *Havlena and Gross*, 1988]

$$c_0(\mathbf{x}, t) = \bar{c}_0(\mathbf{x}) \tag{4}$$

where $\bar{c}_0(\mathbf{x})$ is the spatially variable uplift rate.

Elevations are defined relative to the elevation datum at the outlet of the catchment (e.g., the elevation of the outlet notch). Thus the tectonic uplift rate c_0 is defined as the uplift relative to the elevation of the outlet. As an example, consider a small catchment with an outlet on the floodplain of a very large river. The outlet elevation of the small catchment is dominated by elevation changes in the floodplain in the large river; that is, from the viewpoint of the small catchment the elevation at the outlet is externally imposed and variable in time. In this case, c_0 for the small catchment is the tectonic uplift relative to the floodplain of the large river (i.e., the catchment outlet elevation) not relative to sea level.

Two physically based transport processes are modeled in (1*a*). It is convenient to differentiate between fluvial transport processes that are dependent on discharge and slope and diffusive transport processes that are dependent on slope alone. The most important process over most of the catchment is the continuity term for fluvial sediment transport, the second term in (1*a*). This term encompasses rivers, gullies, rills, and sheet overland flow; only the constants β_1, n_1, m_1, and O_t change. The sediment transport is dependent on discharge and the slope in the steepest downhill direction. Moreover, from (2*c*), the rate of the fluvial transport is a function of whether that point in space is channelized or not. In the model, sediment transport on the hillslopes can be less than that in the channel. This effect is parameterized by the O_t term of (2*c*), where O_t is less than 1. The formulation of this fluvial transport term of (2*b*) is justified in Appendix B. The other important transport term in the elevation evolution equation is a diffusive transport term, the third term in (1*a*). The long-term average of a number of hillslope transport processes can be modeled by use of a spatially constant Fickian diffusion term; the processes include hillslope soil creep, rain splash, and rock slide [*Culling*, 1963; *Dunne*, 1980; *Andrews and Bucknam*, 1987]. Other mass transport mechanisms could also be modeled [*Ahnert*, 1976, 1987] but are not studied here because they require modeling of the regolith depth and thus of the complex chemical and weathering processes that create soil. This was considered outside the scope of the study.

The channelization equation (1*b*) is the equation governing the development of channels and the extension of the networks. Equation (1*b*) is based on one developed by *Meinhardt* [1982] to differentiate the leaf vein cells in a leaf for a biological model of leaf reticulation. It is a convenient equation, based on the phenomenology of channel head extension rather than fundamentally derived from the controlling transport physics at the channel head, which are very complicated and spatially variable. The form of equation (1*b*) causes Y to have two stable attractors, 0 and 1. Typically, the modeling process starts with $Y = 0$ everywhere, representing a catchment with no channels, only hillslopes. If desired, a preexisting channel network can be applied (i.e., $Y = 1$ along the channels). When the value of the channel initiation function a exceeds the channel initiation threshold a_t, the value of $Y = 0$ becomes unstable, and Y goes into transition where it is increasing to $Y = 1$, that is, that spot in the catchment goes into transition from hillslope to channel. When Y reaches a value of 1 it remains there, since the value of $Y = 1$ is stable irrespective of the value of the channel initiation function; channel formation is modeled as a one-way process from hillslope to channel. The role of the channel initiation function is to trigger the channelization process when the threshold is exceeded. The rate at which a point is channelized once the channel initiation threshold is exceeded is governed by the parameter d_t; a large value of d_t results in the channel forming quickly. Equation (1*b*) is a convenient but not exclusive way of parameterizing the abrupt switch from hillslopes to channels in terms of the channel initiation function. Any formulation leading to two stable binary solutions ($Y = 0$; $Y = 1$) and which incorporates the threshold behavior should work similarly. This formulation is believed to adequately simulate the mean position of the channel head, averaging out any stochastic advance and retreat of the channel head that may occur over short time scales.

Four properties of the channel initiation function were found to be necessary before network development would occur. Without these properties the channels, represented by the 0-1 state, either did not extend, did not form binary branched networks, or simply formed disconnected "blobs." In the discussion below the channel initiation function may be considered to be either overland flow velocity or groundwater seepage velocity, both positively correlated with discharge per unit width and slope. This assertion will be justified later in this paper.

The first of these four networking conditions was that regions of elevated channel initiation function must be formed on the hillslopes around the advancing channel head. These regions resulted from the localized high slopes and the convergence of the flow paths around the channel head (Figure 1*a*). These in turn resulted from preferential erosion in the channels. Simply put, channels must erode faster than hillslopes, and this condition is required for a channel head to advance.

The second condition for network formation was that the region of elevated channel initiation function around the channel head must move upstream with the channel head as it advances. The capturing of the flow around the channel head (Figure 1*b*) ensures this. In this way, discharges per unit width on the hillslope downstream of the channel head are diminished, relative to those in front of the channel head, reducing the channel initiation function. This condition allows the channel to grow linearly rather than in a blob-like form.

The third conditions was that advancing channel heads should "repel" each other and that growing channel heads should be repelled by a watershed or catchment boundary. This condition ensures that the resulting network is space filling, a commonly assumed, if not totally verified, feature of channel networks [*Abrahams*, 1984; *Tarboton et al.*, 1988, 1989]. The repulsion of advancing channel heads results from an interaction between the drainage patterns and erosion (Figure 1*c*). Everything else being equal, the region between the growing channel heads (region A) which has lower discharges per unit width will have a slower rate of change of elevation than the region outside the advancing

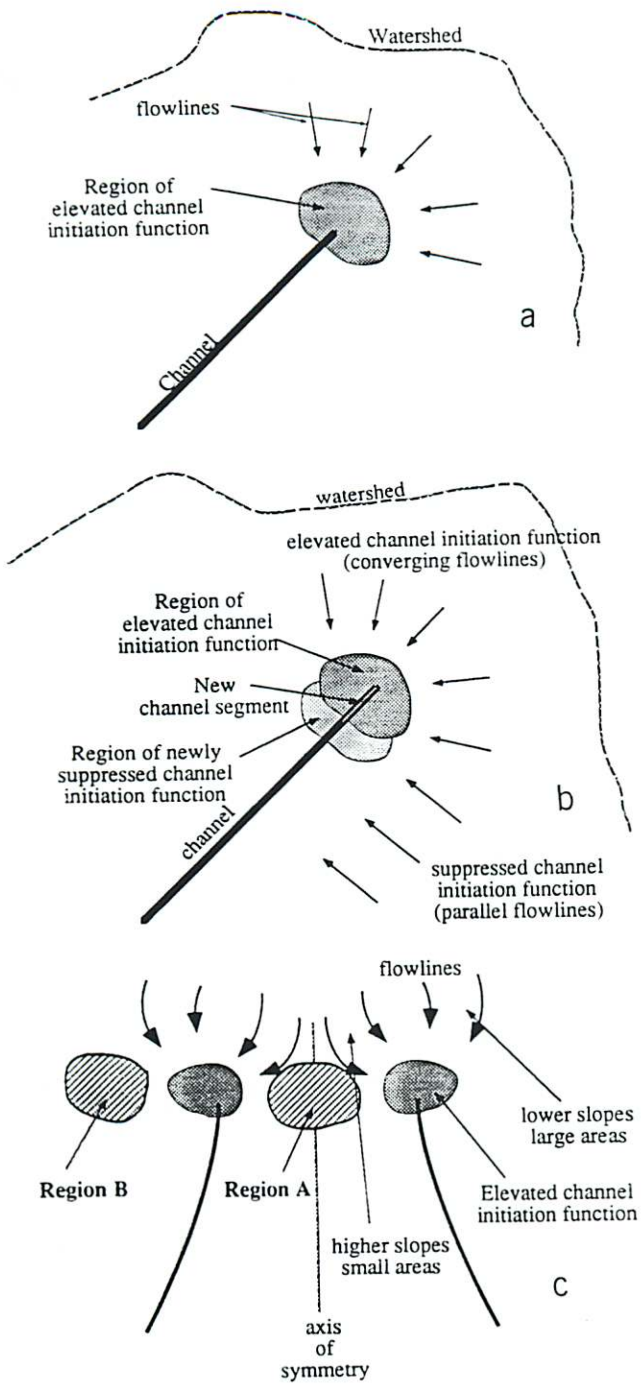

Fig. 1. (*a*) Localization of channel initiation function around channel head. (*b*) Translation of channel initiation function region with channel head activator. (*c*) "Repulsion" of channel heads.

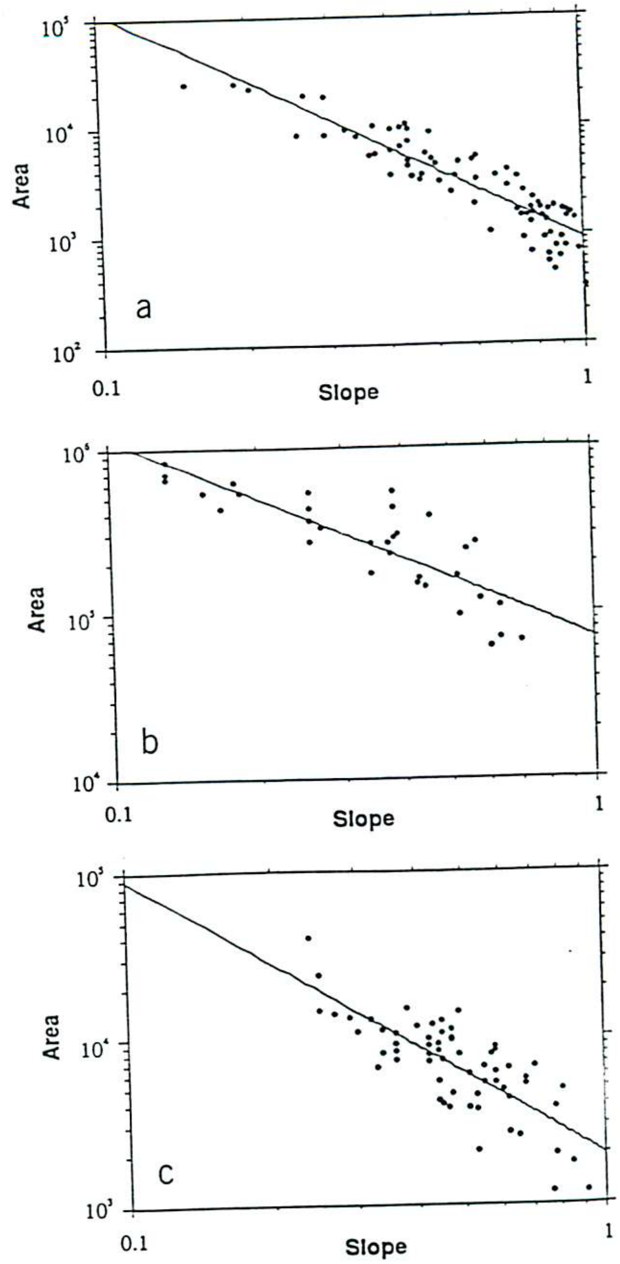

Fig. 2. Source area plotted against valley slope, or slope at channel heads. for (*a*) Coos Bay, Oregon, (*b*) Sierra Nevada, California, and (*c*) Marin County, California. The line is the channel initiation mechanism (equation (11)). (The data were digitized from graphs courtesy of W. Dietrich.)

channel heads which has larger discharges per unit width (region B). Hence, although slopes are highest between the advancing channel heads (region A), for typical channel initiation functions (see below) the highest channel initiation function is typically in region B so that advancing channel heads repel each other. Repulsion from the boundaries follows similarly, with the boundary being a watershed (i.e., zero slope perpendicular to the boundary, equation (2*g*)), so that an image channel may be postulated.

The fourth condition is that the process that produced the elevated channel initiation function around the channel head and lower values elsewhere must be localized around the channel head. The limited length scale of this process ensures that new channel heads can be created laterally off existing channels, away from existing channel heads.

PHYSICAL JUSTIFICATION OF THE CHANNEL INITIATION FUNCTION

The generic equation used to represent the channel initiation function in (2*a*) is

$$a = \beta_5 q^{m_5} S^{n_5} \quad (5)$$

TABLE 1. Sample Channel Initiation Functions

Mechanism	Governing Equation*	m_5/n_5
1. overland flow velocity (per unit width)	$v = \left[\frac{1}{n^{3/5}}\right] q^{0.4} S^{0.3}$	1.33
2. overland flow† velocity (triangular channel)	$v = \left[\frac{a_1}{4(1 + a_1^2) n^3}\right]^{0.25} Q^{0.25} S^{0.3}$	0.67
3. overland flow shear stress (/unit width)	$\tau = [\gamma n^{3/5}] q^{0.6} S^{0.7}$	0.86
4. overland flow shear stress (triangular channel)	$\tau = \left[\frac{n a_1 \gamma^{8/3}}{4(1 + a_1^2)^{1/2}}\right]^{3/8} Q^{0.375} S^{0.813}$	0.46
5. groundwater stream sapping	$\frac{dH}{dx} = \left[\frac{1}{Kh}\right] q$	∞

*For notation, see the text.
†a_1 is the sideslope of the triangular channel.

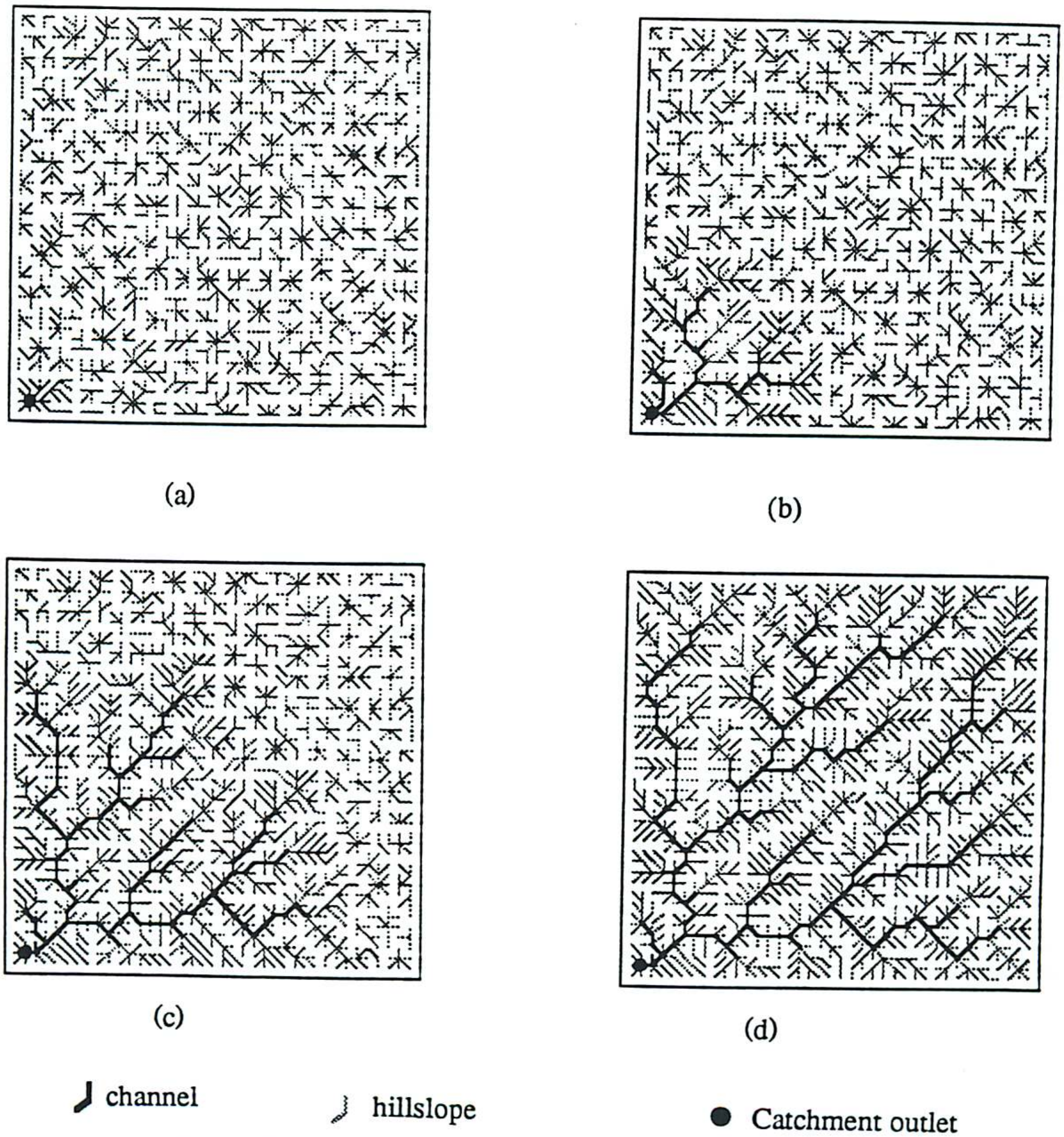

Fig. 3. Simulated channel networks and hillslope flow directions with time.

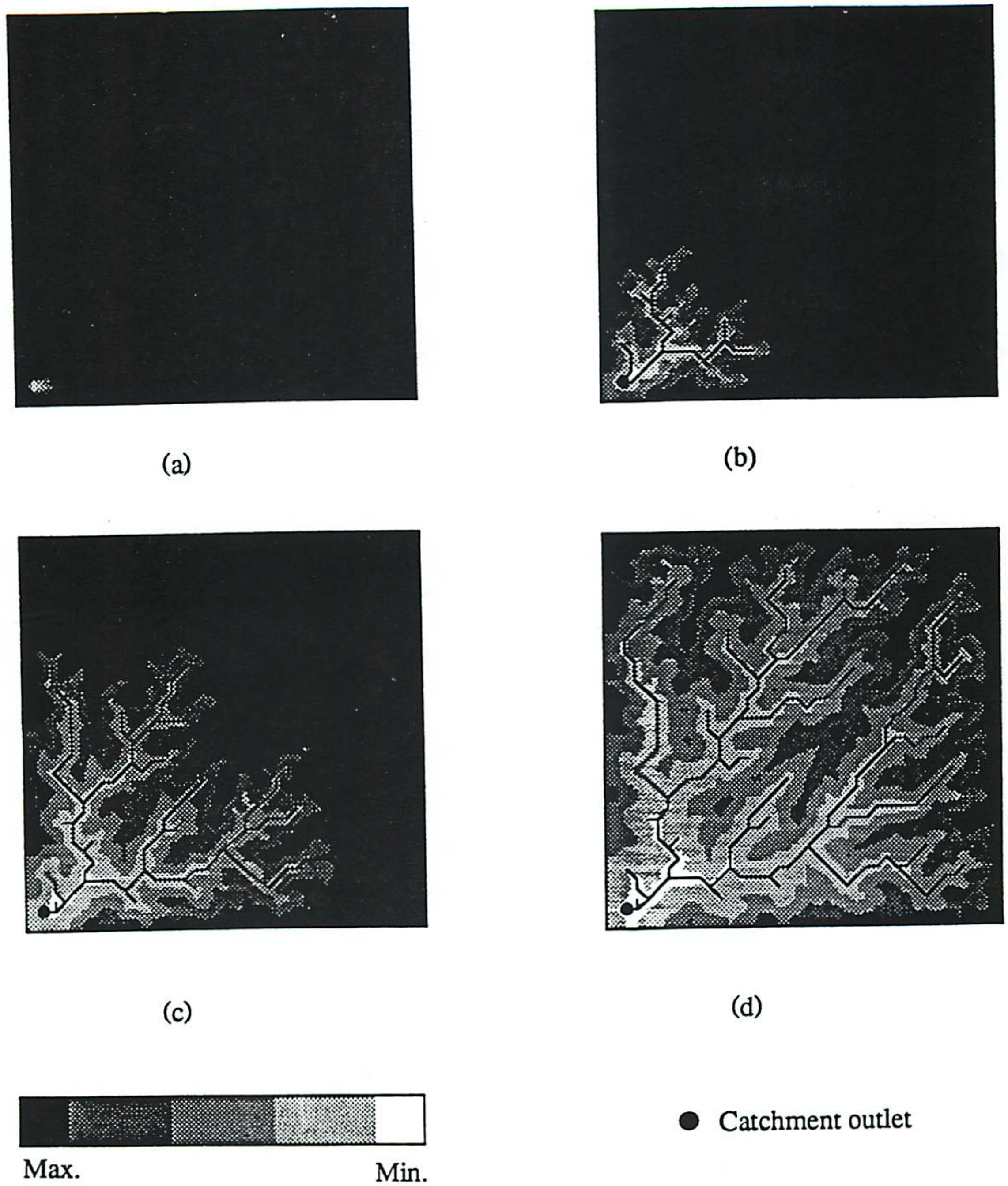

Fig. 4. Simulated channel network and elevation with time.

The purpose of the channel initiation function in the channelization indicator equation (1b) is to trigger the transition from hillslope to channel when the channel initiation threshold at that point is exceeded. The channel initiation function reflects both large-scale and small-scale channel extension processes. The large-scale processes are the hillslope scale erosion processes such as the discharge and slope at the channel head. The small-scale processes, smaller than the spatial resolution of the model, are those related to the geometry of the channel head. These geometry effects are explicitly incorporated into the single coefficient β_5, as outlined below.

A number of different physical processes that can trigger channel head advance will be examined below. One of the most common criteria for the design of erosion works is overland flow velocity. If the wide channel assumption is made so that hydraulic radius R is equal to flow depth y, and the wetted perimeter P is equal to the channel width w, then Manning's equation can be written as

$$v = \frac{y^{2/3} S^{1/2}}{n} \tag{6}$$

and the discharge Q for a wide channel of width w can be written as

$$Q = \frac{y^{5/3} S^{1/2} w}{n} \tag{7}$$

Combining these equations yields

$$v = \beta_5 q^{0.4} S^{0.3} \tag{8}$$

where $\beta_5 = [n^{3/5}]^{-1}$.

The width w may be the width of the upstream face of the channel head so that Q/w is the discharge per unit width on the hillslope directly upstream of the channel head. Alternatively, if the hillslope hollow upstream of the channel head has concentrated the flow into a rill, the appropriate width may be the rill width. The model does not determine this width; it must be determined a priori in combination with the other unknowns. This may be quite difficult in the field.

Similar expressions for velocity in a triangular channel, and other channel geometries, may be derived, and the exponents are different [*Willgoose et al.*, 1989]. Bottom shear stress can also be used as the threshold criteria. Results are summarized in Table 1.

Dunne [1969] proposed a conceptualization of a groundwater process where groundwater stream tubes converged onto a seepage face at a channel head, causing channel side wall erosion. This conceptualization of gully advance is supported by other fieldwork [*Priest et al.*, 1975]. *Dunne* [1989] suggested a threshold on the hydraulic gradient above which erosion at the seepage force will occur by piping.

$$\left(\frac{dH}{dx}\right)_{\text{threshold}} = (\gamma - 1)(1 - n) \tag{9}$$

where dH/dx is groundwater hydraulic gradient at the seepage face, γ is specific gravity of the sediment material, and n is porosity.

Using Darcy's law for groundwater flow at the seepage face, this can be reformulated as

$$\frac{dH}{dx} = \beta_5 q \tag{10}$$

where

$\beta_5 = 1/Kh$;
K hydraulic conductivity;
h height of the seepage face;
q discharge/unit width.

Thus the general formulation of (5) is equally applicable to channel formation due to surface-dominated or groundwater-dominated hydrologic processes.

There exists some experimental evidence to support a threshold-based channel initiation mechanism, dependent on discharge and slope, as proposed above. The idea of channel extension occurring when some threshold is exceeded is not new. For instance, *Dietrich et al.* [1986] and *Montgomery and Dietrich* [1988] proposed a channel head advance mechanism similar in concept to that here. Their mechanism, argued on the basis of slope stability and landsliding, initiated channel growth when a function dependent on the hillslope stability exceeded a threshold.

Furthermore, *Patton and Schumm* [1975] and *Begin and Schumm* [1979] examined the slopes and area contributing to a point in the hillslopes and channels. Patton and Schumm found that the data for channels and hillslopes were significantly different at the 1% level, suggesting that a threshold separates the channel and the hillslope regimes. *Begin and Schumm* [1979, p. 349] found for a given contributing area that "above a certain threshold slope the probability of valley incision is largely increased."

The notion of a threshold on channel initiation does not necessarily contradict the channel stability concept of *Smith and Bretherton* [1972], where channels form when a small surface nick grows unstably. Recently, *Loewenherz* [1990]

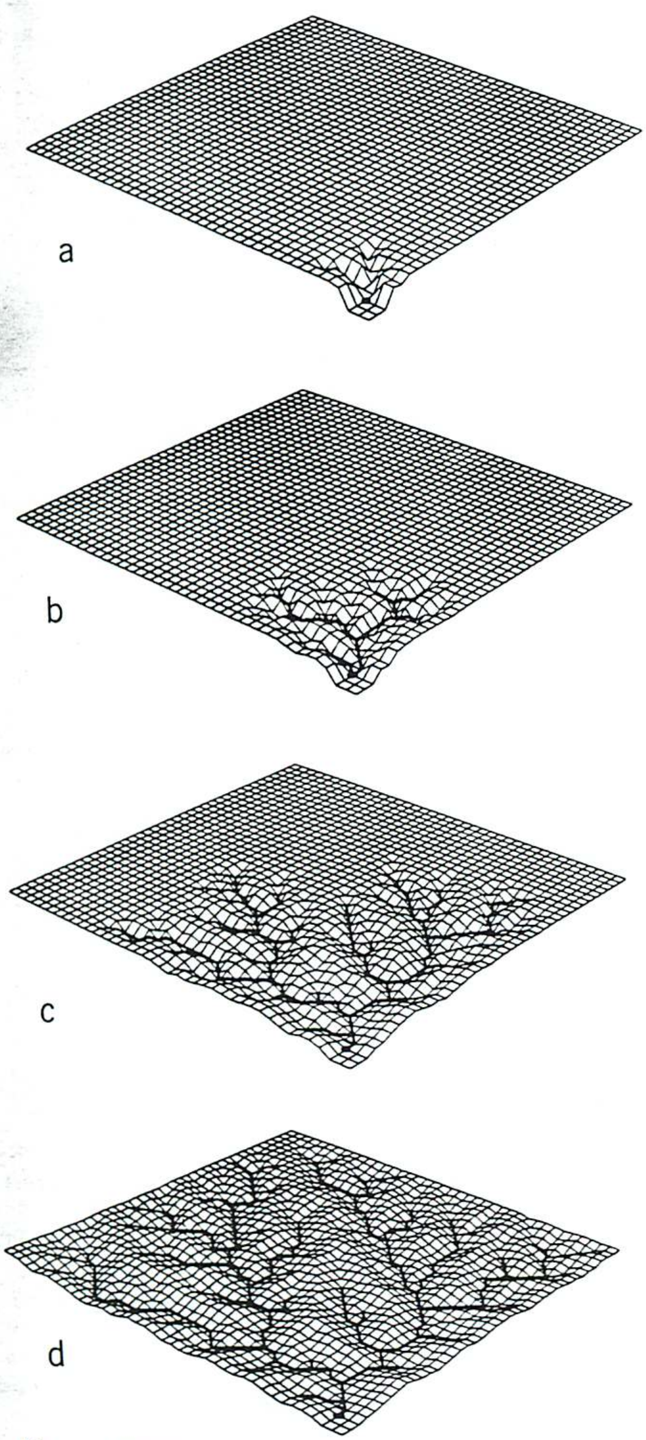

Fig. 5. Isometric view of channel network and hillslope with time: (*a*) t = 500, (*b*) t = 2000, (*c*) t = 6000, (*d*) t = 13,000. Times are nondimensional.

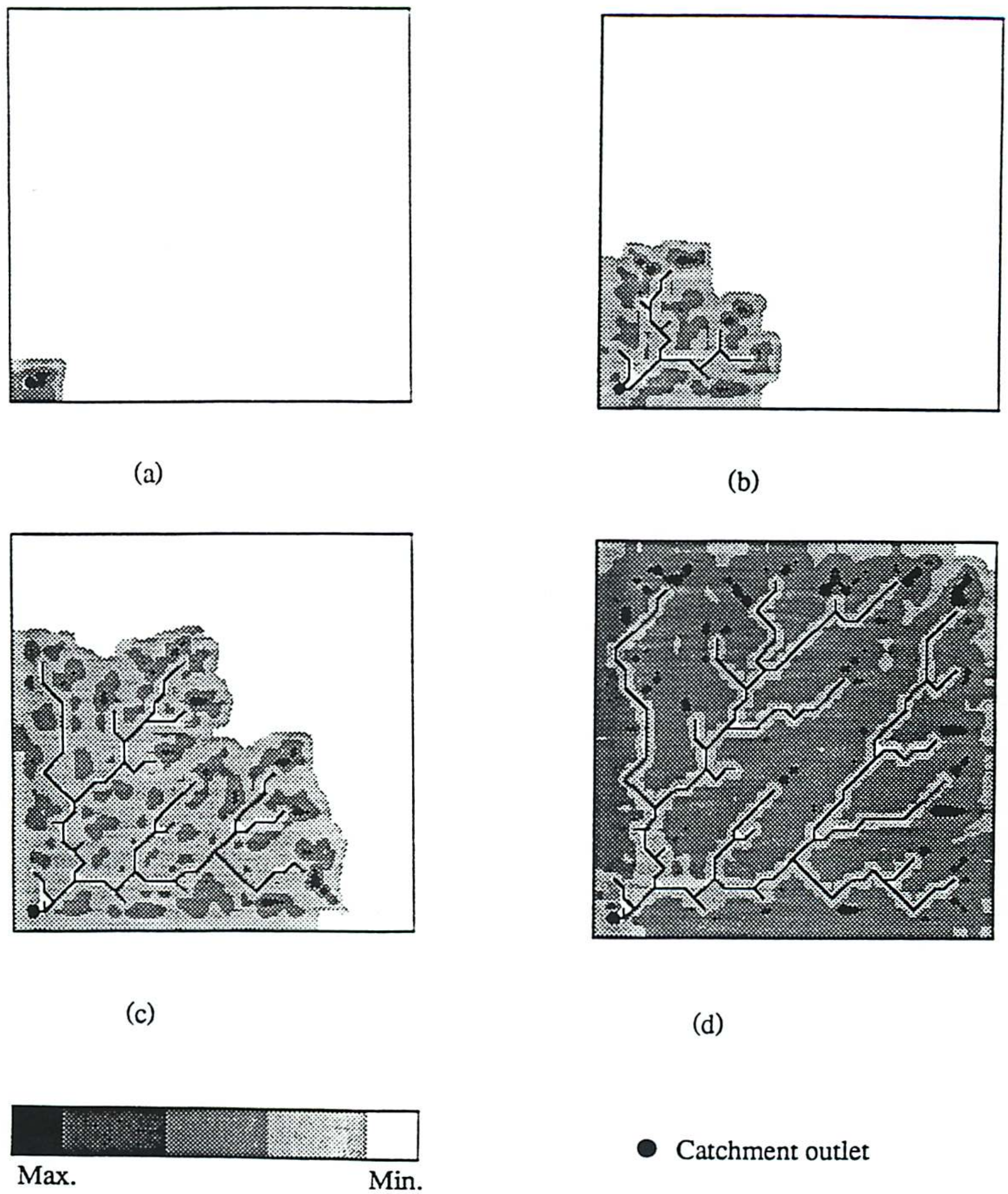

Fig. 6. Simulated channel network and overland flow velocities with time.

concluded that the Smith and Bretherton analysis will lead to a system of rills spaced at an infinitesimal distance apart. This contradicts the observations that incisions and streams are separated by finite distances. *Loewenherz* [1990] resolved the stability conditions by introducing microscale effects that damp the growth of small wavelengths, effectively introducing a scale into the problem. This is compatible with a threshold concept. Furthermore, it is commonly observed that gully extension occurs after land clearing (increasing runoff or decreasing erosional resistance), a phenomenon that can be explained by a threshold concept based on runoff and erodability and not necessarily with stability analysis. We suggest that the Smith-Bretherton criterion determines the most upstream point to which the channel head may advance but that otherwise the channel head position is determined by the threshold.

The proposed relationship between discharge and slope in the channel initiation function is also supported by experimental evidence. A channel head will stop advancing when the channel initiation function in the hillslope just upstream of the channel head falls below the threshold. Thus at the channel head the channel initiation function will be equal to or less than the threshold, so that at the channel head,

$$A^{m_3 m_5/n_5} S \le \left(\frac{a_t}{\beta_5 \beta_3^{m_5}} \right)^{1/n_5} \tag{11}$$

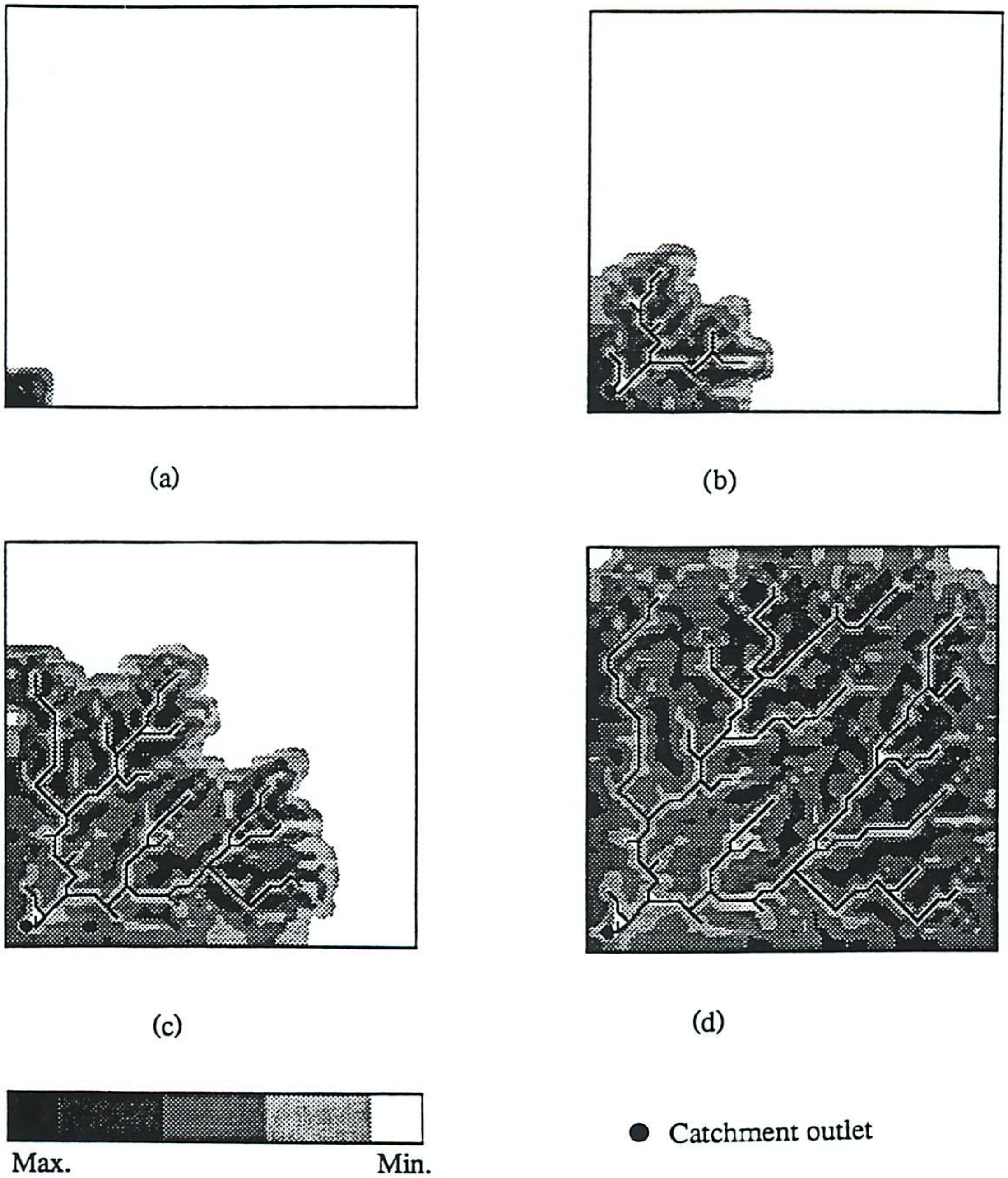

Fig. 7. Simulated channel network and hillslope slopes with time.

Montgomery and Dietrich [1988, 1989] collected data for gully heads in three regions of California and Oregon. They measured the area contributing to the channel head and the slope of the valley immediately upstream of the channel head. No one gully head advance mechanism was dominant over all the regions, and gully advance was attributed to a variety of different mechanisms including localized landsliding, groundwater stream sapping, and overland flow. No attempt was made to classify the data on the dominant mechanism at the gully head. The equality in (11) was fitted to these data, and the results are presented in Figure 2. The fitted values of $m_3 m_5/n_5$ are consistent with proposed mechanisms for channel head advance [*Willgoose et al.*, 1989], and the fit to the data of (11) is quite satisfactory. Moreover, the fitted values of the right-hand side constant in (11) are consistent with the regional trends in mean annual rainfall and thus the runoff β_3. It must be stated, however, that in many cases the observed channel head processes were different than those discussed here. This suggests that the other channel initiation processes noted above may also be formulated by a threshold criteria of the form of (11). The small amount of scatter in the data of Montgomery and Dietrich either suggests that the right-hand side of (11) is constant across processes or that there is only one dominant channel extension process in the field, not the several as suggested. This issue will be further discussed in an accompanying paper [*Willgoose et al.*, this issue (*b*)].

Note from (11) that if the ratio $m_3 m_5/n_5$ is the same (with everything else the same) for two basins, then the area-slope relationship of the two systems is similar (has the same slope

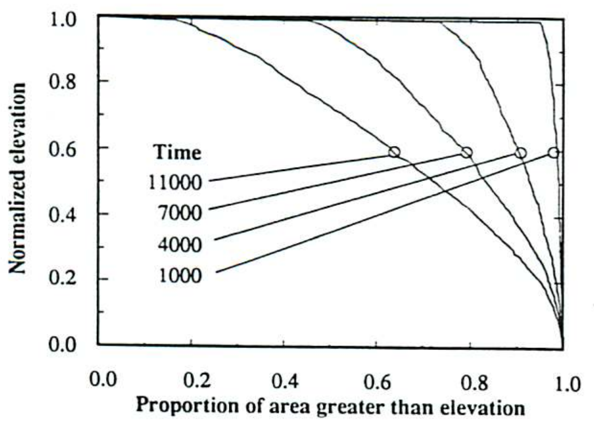

Fig. 8. Hypsometric curve with time for simulated catchment.

in a log-log plot). The drainage density, though, depends on the absolute values of m_5 and n_5, since it depends on the threshold relationship on the right-hand side of (11). For two networks to be exactly the same the spatial distribution of locations where the threshold is exceeded throughout the catchment's history must be the same.

Sample Results

This section presents some sample results of the application of the computer model documented in Appendix A, based on the theory described above. This section is not intended to be a comprehensive consideration of all aspects of the model and the simulations; that will be the subject of an accompanying paper [*Willgoose et al.*, this issue (*a*)]. Rather, following a single simulation through time, typical characteristics of the generated catchments will be described.

The results of the simulation presented here are typical of the large number of simulations that have been performed. Figures 3–6 show the spatial distribution of various catchment properties for selected times. Figure 3 shows the simulated channel network. The initial surface is a flat plane with a very minor random elevation perturbation (0.25% of the initial notch height). Thus initially much of the hillslope self-drains to pits, where the flow, but not sediment, is assumed to infiltrate to groundwater at the lowest point. Figure 3 demonstrates the headward growth of the channels from the initial seed on the bottom left-hand corner of the grid. The directions of overland flow are also shown, and they demonstrate the convergence of flow directions on the hillslopes around the channel heads illustrated in Figure 1. As the network extends, the lower valleys surrounding the channels capture the self-draining portions of the hillslope. The network resulting from lateral branching is qualitatively similar to branching in the stream-sapping hypothesis of *Dunne* [1969], and the pattern of future channel branching is likewise mirrored by the current pattern of hillslope flow directions.

As the network grows, it erodes valleys along the channels because of the preferential erosion in the channels compared with the hillslopes. Figure 4, contours of elevation, clearly shows this. These valleys result in the preferred hillslope flow directions being toward the channel network so that there are high velocities around the channel head. An alternative view of this valley formation with time is given by Figure 5, which is an isometric view of elevations within the catchments.

The network growth process is dominated by the spatial distribution of the channel initiation function on the hillslopes. The channel initiation function in this example is overland flow velocity, as described by (8). Figure 6 shows the spatial distribution of velocity on the hillslopes. This figure demonstrates that the regions of high velocity are concentrated around the channel heads and move with the advancing channel heads. In particular, Figure 6*c* shows that the highest peaks of the channel initiation function are at the growing channel heads and that other peaks within in the interior parts of the network are considerably lower. This is consistent with the first two of the networking conditions discussed at the start of the paper. At later times the channels, and the regions of high velocity, are relatively uniformly spaced, which is consistent with the idea of space-filling networks discussed by *Abrahams* [1984] and related to the third networking condition discussed in the physical model section.

Contours of hillslope slope are provided in Figure 7. The most interesting characteristic of this plot is that the steepest slopes do not occur around the advancing channel head. The steepest slopes are on the laterally draining valley sides; the slopes draining down the valley to the channel heads are quite low by comparison. However, the channel initiation function is high at the channel heads, even though the

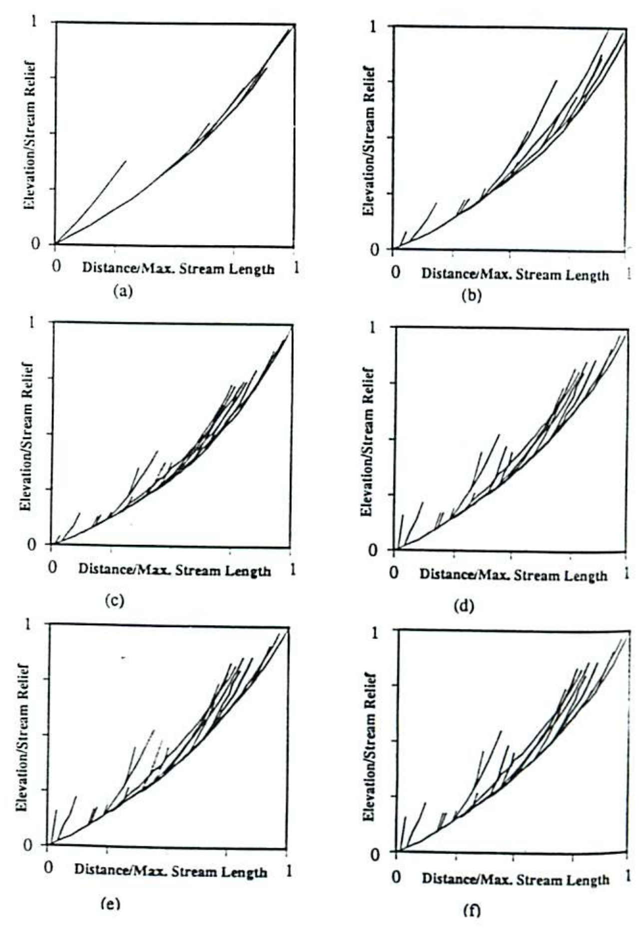

Fig. 9. Longitudinal elevation of all channels with time: (*a*) t = 2000, (*b*) t = 6000, (*c*) t = 13,000, (*d*) t = 25,000, (*e*) t = 35,000, and (*f*) t = 60,000.

corresponding slopes are low, because of the compensating effect of large contributing areas. In Figure 3 the hillslopes contributing to the channel head are long, relative to those draining laterally to the channel, so that the contributing area per unit width at the channel heads is larger than that draining laterally. Thus in the channel initiation function of (2a) the increased area contribution overwhelms the decreased slope contribution. Figure 7 shows that as the catchment evolves with time the highest slopes are in the upstream reaches of the catchment, with lower slopes downstream. These lower slopes result from the hillslope erosion that has taken place in the older, root sections of the catchment of the bottom left-hand corner (see Figure 4).

Figure 8 gives the hypsometric curves for the catchment. The shape and trends of this curve with time are consistent with the interpretation of field data proposed by *Schumm* [1956]. Figure 9 shows the elevations of all the streams, normalized against both distance and elevation, for a variety of times both before and after the network has stopped growing. The curvature of the profile is reasonable and consistent with observed data. Table 2 lists some sample statistics for the catchment for the time at which the network stops growing.

In conclusion, a sample simulated catchment has been shown, and the characteristics of a typical simulation have been discussed. The networks that are generated are qualitatively realistic both in planar and elevation profile properties. The drainage directions on the hillslopes are shown to be consistent with channel network growth hypotheses of previous researchers and consistent with the stream-sapping, headward growth mechanism.

CONCLUSIONS

This work developed a physically based model of channel network growth and hillslope evolution. The model simulates the long-term changes in elevation within the catchment and the consequent effect on channel network growth and hillslope form. The changes in elevation are modeled by continuity equations for flow and sediment transport; elevation changes result from local imbalances in the sediment transport. A channelization mechanism, called the channel initiation function, which is nonlinearly dependent on discharge and local slope is used. It is the spatial distribution of the channel initiation function around the channel head that governs where and whether the channel head advances; a channel extends if the channel initiation function exceeds a threshold. A central component of the model is that erosion in the channels takes place at a faster rate than in the hillslope. This preferential erosion in the channels results in convergence of the flow on the hillslopes toward the channels. It is this convergence of flow that triggers channel head advance. Thus the interaction between the hillslopes and the channels over long time scales is central to the final form of the channel network and the hillslopes.

A sample simulated catchment was examined. This simulation demonstrated the process by which catchments grow and develop their observed form. It was also demonstrated that the physics of the evolution of the catchment and network growth are consistent with observed characteristics of field catchments.

TABLE 2. Sample Statistics of Simulated Basin

Statistic	Value
R_b	5.20
R_s	1.73
K	1.78
D'_d	6.80
R_l	2.85
R_A	6.61
ε_1	2.25
magnitude	22
Mean catchment relief	9.90
Mean hillslope relief	4.18
Mean stream relief	6.41
Mean hillslope slope	1.92

R_b, Horton's bifurcation ratio; R_s, Horton's slope ratio; R_l, Horton's length ratio; D'_d, nondimensional drainage density $D_dA^{1/2}$; A, catchment area; ε_1, number of streams of order $j - 1$ flowing into streams of order j, for any j, as defined by *Tokunaga* [1978]; K, ratio $\varepsilon_k/\varepsilon_{k-1}$ where ε_k is the number of streams of order $j - k$ flowing into streams of order j, for all j, as defined by *Tokunaga* [1978].

APPENDIX A: NUMERICAL SOLUTION TECHNIQUE FOR THE GOVERNING EQUATIONS

Equations (1a) and (1b) are solved on a two-dimensional rectangular grid with irregular boundaries allowed. To begin the calculations, initial elevations are assigned to the nodes, and the catchment is assigned an initial pattern of channelization. Using this elevation information, a drainage direction at each node is determined. A node may only drain into one of the eight nodes directly adjacent to it. All flow in a node drains in the steepest downslope direction. Contributing area to a node is determined by analyzing the drainage directions.

Elevation changes result from imbalances in sediment transport. The balance of sediment transport at a node is determined by evaluating the sediment transport into that node and subtracting the sediment transport out of that node. The sediment transport equation (2b) is evaluated at every node so that the rate of change of elevation at a node j due to fluvial sediment transport alone is

$$\frac{\partial z_j}{\partial t} = \frac{1}{\rho_s(1-n)\Delta x \Delta y} \sum_{i=1}^{N} I_{ij} f(Y_i) Q_i^{m_1} S_i^{n_1} \qquad \text{(A1)}$$

where

$I_{ij} = 1$ if node i drains into node j;
$I_{ij} = 0$ if node i does not drain into node j;
$I_{ij} = -1$ when $i = j$;
N number of nodes in the grid;
Δx, Δy grid spacing in the x and y directions;
$f(Y_i) = \beta_1$ when $Y_i = 1$ (channel);
$f(Y_i) = \beta_1 O_t$ when $Y_i = 0$ (hillslope).

Note that the interpretation of the discharge here is different from that expressed in the governing equations (1) and (2). Here Q_i is the total discharge through the node i on a grid Δx by Δy which is related to the discharges in the governing equations (for flow in the y direction as an example) as

$$Q_{iy} = \int_{i-\Delta x/2}^{i+\Delta x/2} q_y \, dx \qquad \text{(A2)}$$

The idea of $Y = 0$ representing hillslope and $Y = 1$ representing channels is only approximate. In particular, at an advancing channel head there is a period of time when the hillslope is in transition from hillslope to channel; that is, Y is between 0 and 1. Points intermediate between hillslope and channel have sediment transport properties that are intermediate between that for hillslope and that for channel, though simulations indicated that the model is insensitive to the exact form. The adopted transition was a linear transformation from hillslope to channel sediment transport rate.

$$f(Y) = \beta_1 O_t \qquad Y \le 0.1\alpha \text{ (hillslope)}$$

$$f(Y) = \beta_1\left[O_t + (1 - O_t)(Y - 0.1\alpha)\frac{Y}{1 - 0.1\alpha}\right] \qquad 0.1\alpha \le Y < \alpha \text{ (transition)}$$

$$f(Y) = \beta_1 \qquad Y \ge \alpha \text{ (channel)} \tag{A3}$$

where α is a model parameter greater than 1.

The Fickian diffusion term in (1a) is evaluated in space by a five-point centered finite difference approximation so that

$$D_z\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)_{(i,j)} \approx D_z\left(\frac{z_{i+1,j} + z_{i,j+1} + z_{i-1,j} + z_{i,j-1} - 4z_{i,j}}{\Delta x \Delta y}\right) \tag{A4}$$

where D_z is diffusivity and $z_{i,j}$ is elevation at the node with the (x, y) coordinates equal to (i, j).

Equations (A1) and (A4) are solved in time by an explicit two-point predictor-corrector scheme (see, for example, *Acton* [1970]). Equation (A1) is stiff because of the large variation of A (and thus Q) over a catchment; area varies from 1 to NM nodes for an N by M region. The rates of change of elevation for large areas are high, those for small areas (e.g., hillslopes) are low. To capture the details of elevation changes for both the large and small areas, and maintain numerical stability on the basis of the Courant number [*Willgoose*, 1989], very small time steps are required.

The preferred solution scheme uses a nonlinear extrapolation for the predictor step and nonlinear interpolation for the corrector step. This scheme explicitly addresses the time scale problems noted above by using a nonlinear extrapolation for elevation that is an approximate solution to (A1). The solution technique is thus

$$\mathbf{z}^p(t_0 + \Delta t) = \mathbf{z}(t_0) + \Delta\mathbf{z}^p \tag{A5}$$

$$\mathbf{Y}^p(t_0 + \Delta t) = \mathbf{Y}(t_0) + \Delta\mathbf{Y}^p \tag{A6}$$

$$\mathbf{z}^c(t_0 + \Delta t) = \mathbf{z}(t_0) + 0.5(\Delta\mathbf{z}^p + \Delta\mathbf{z}^c) \tag{A7}$$

$$\mathbf{Y}^c(t_0 + \Delta t) = \mathbf{Y}(t_0) + \frac{\Delta t}{2}\left(\left.\frac{\partial \mathbf{Y}}{\partial t}\right|_{\mathbf{z} = \mathbf{z}^p(t_0 + \Delta t), \mathbf{Y} = \mathbf{Y}^p(t_0 + \Delta t)} + \left.\frac{\partial \mathbf{Y}}{\partial t}\right|_{\mathbf{z} = \mathbf{z}(t_0), \mathbf{Y} = \mathbf{Y}(t_0)}\right) \tag{A8}$$

where $\Delta\mathbf{z}^p$ is the predicted change in $\mathbf{z}$, relative to $\mathbf{z}(t_0)$, over the time step Δt, based on the states at time $t_0[\mathbf{z}(t_0), \mathbf{Y}(t_0)]$, and $\Delta\mathbf{z}^c$ is the corrected changes in $\mathbf{z}$, relative to $\mathbf{z}(t_0)$, based on the states at time $t_0[\mathbf{z}(t_0), \mathbf{Y}(t_0)]$ and the predicted states at time $t_0 + \Delta t$ $[\mathbf{z}^p(t_0 + \Delta t), \mathbf{Y}^p(t_0 + \Delta t)]$.

The following derivation develops the nonlinear extrapolation method that is an approximation to the exact solution of (A1). Assume that fluvial transport dominates diffusive transport. Consider fluvial sediment continuity at a node j where the elevation at all surrounding nodes are fixed with time:

$$\frac{\partial z_j}{\partial t} = \frac{1}{1 - n}\left[\sum_{i \ne j} f(Y_i) I_{ij} Q_i^{m_1}\left(\frac{z_i - z_j}{l_{ij}}\right)^{n_1} - f(Y_j) Q_j^{m_1}\left(\frac{z_j - z_k}{l_{jk}}\right)^{n_1}\right] \tag{A9}$$

where l_{ij} is the distance between nodes i and j and k is the node that node j drains into.

As time goes to infinity and elevations go to equilibrium, then

$$\frac{1}{1 - n}\left[\sum_{i \ne j} f(Y_i) I_{ij} Q_i^{m_1}\left(\frac{z_i - z_j^*}{l_{ij}}\right)^{n_1} - f(Y_j) Q_j^{m_1}\left(\frac{z_j^* - z_k}{l_{jk}}\right)^{n_1}\right] = 0 \tag{A10}$$

For the nonlinear extrapolation of elevations we need to know (1) the value of the equilibrium elevation z_j^* and (2) the rate at which $z_j(t)$ tends to z_j^*.

It can be shown [*Willgoose et al.*, 1989] that (A9) is approximated

$$\frac{\partial z_j}{\partial t} = \left.\frac{\partial z_j}{\partial t}\right|_{t = t_0} \frac{(z_j(t) - z_j^*)^{n_1}}{(z_j(t_0) - z_j^*)^{n_1}} \qquad t \ge t_0 \tag{A11}$$

An important property of this equation is that it is asymptotically correct for both $t = t_0$ and $t = \infty$ so that many of the stiffness problems of (A1) are obviated. Solution of this equation yields the solution for $z_j(t_0 + \Delta t)$, given the equilibrium elevation z_j^*, as

$$z_j(t_0 + \Delta t) = z_j^* + [z_j(t_0) - z_j^*]^{n_1/(n_1 - 1)}\left[z_j(t_0) - z_j^* + \left.\frac{\partial z_j}{\partial t}\right|_{t = t_0}(1 - m)\Delta t\right]^{(1 - n_1)^{-1}} \tag{A12}$$

To determine z_j^*, $\partial z_j/\partial t$ is approximated by a Taylor series (expanded to linear terms), around the elevation at time t_0, $z_j(t_0)$, so that

$$\frac{\partial z_j}{\partial t} \approx \left.\frac{\partial z_j}{\partial t}\right|_{z_j = z_j(t_0)} + [z_j - z_j(t_0)]\left[\frac{\partial}{\partial z_j}\left(\frac{\partial z_j}{\partial t}\right)\right]\Bigg|_{z_j = z_j(t_0)} \tag{A13}$$

Solving this equation for the equilibrium solution $z_j = z_j^*$, when $\partial z_j/\partial t = 0$, yields the estimate of z_j^* based on elevations at $t = t_0$:

$$z_j^* = \frac{z_j(t_0)\left[\dfrac{\partial}{\partial z_j}\left(\dfrac{\partial z_j}{\partial t}\right)\right]\Bigg|_{z_j = z_j(t_0)} - \left.\dfrac{\partial z_j}{dt}\right|_{z_j - z_j(t_0)}}{\left[\dfrac{\partial}{\partial z_j}\left(\dfrac{\partial z_j}{\partial t}\right)\right]\Bigg|_{z_j = z_j(t_0)}} \tag{A14}$$

The derivative with respect to z_j in this equation can be determined from (A9) and is given by

$$\frac{\partial}{\partial z_j}\left(\frac{\partial z_j}{\partial t}\right) = \frac{1}{1-n}\left[\sum_{\substack{i \\ i \neq j}} \frac{f(Y_i) I_{ij} Q_i^{m_1}}{l_{ij}} \left(\frac{z_i - z_j}{l_{ij}}\right)^{n_1 - 1} + \frac{f(Y_j) Q_j^{m_1}}{l_{jk}} \left(\frac{z_j - z_k}{l_{jk}}\right)^{n_1 - 1}\right] \quad \text{(A15)}$$

APPENDIX B: PHYSICAL JUSTIFICATION OF THE SEDIMENT TRANSPORT EQUATION

The sediment transport formula given in (2*b*) is of the form

$$q_s = \beta_1 q^{m_1} S^{n_1} \quad \text{(B1)}$$

A sediment transport equation of this form has been used by geomorphologists in previous work [e.g., *Smith and Bretherton*, 1972] and may be obtained from the Einstein-Brown equation, a commonly accepted fluvial sediment transport formula, with a minimal number of simplifying assumptions. In addition, it will be shown how the Einstein-Brown equation, an instantaneous sediment transport relation, can be converted into a mean temporal sediment transport relation for long time scales. It will be shown that the simple form of (B1) is maintained after temporal averaging.

The Einstein-Brown equation is expressed in terms of a nondimensional sediment transport ϕ and a nondimensionalized shear stress $1/\psi$. *Vanoni* [1975] gives the governing equation as

$$\phi = 40\left(\frac{1}{\psi}\right)^3 \quad \text{(B2)}$$

where

$$\phi = \frac{q_s}{\rho_s F_1 \sqrt{g(s-1)d_s^3}} \quad \text{(B3)}$$

$$\frac{1}{\psi} = \frac{\tau_0}{\rho g(s-1)\, d_s} \quad \text{(B4)}$$

$$F_1 = \sqrt{\frac{2}{3} + \frac{36\nu^2}{g d_s^3 (s-1)}} - \sqrt{\frac{36\nu^2}{g d_s^3 (s-1)}} \quad \text{(B5)}$$

and the notation used is

- q_s sediment discharge, mass/time/(unit width);
- s specific gravity of sediment;
- ρ, ρ_s density of water and sediment, respectively;
- d_s a representative diameter for the sediment particle (normally d_{50}, the 50th percentile diameter, is used);
- g acceleration due to gravity;
- τ_0 γRS, which is bottom shear stress;
- R hydraulic radius;
- S bed slope;
- ν kinematic viscosity of water.

If the sediment is considered homogeneous throughout the catchment, this equation may be simplified to yield

$$q_s = F_2 (RS)^p \quad \text{(B6)}$$

where $p = 3$ for the Einstein-Brown equation, and

$$F_2 = 40 \rho_s F_1 \sqrt{g(s-1)d_s^3}\left[\frac{1}{(s-1)\, d_s}\right]^3$$

Equation (B6) is not in the form of (B1). The equation must be reformulated so that it is dependent on discharge per unit width q rather than the hydraulic radius. A number of different channel geometries have been examined [*Willgoose et al.*, 1989], including (1) a wide channel with uniform depth across the cross section, (2) overland flow/unit width, (3) a triangular channel with side slopes a_1, and (4) a general channel cross section of the form $y = a_1|x|^{b_1}$, where a_1 and b_1 are variable.

The simplest case, a wide channel, will be used to illustrate the techniques involved. Using Manning's equation for discharge (equation (7)), noting that for a wide channel $R = y$, the governing sediment equation for the wide channel is

$$q_s = F q^{3p/5} S^{7p/10} \quad \text{(B7)}$$

where $F = F_2 n^{3p/5}$.

The multiplicative constant F is dependent, in a well defined way, on flow geometry and sediment characteristics. F is constant with respect to q because the wetted perimeter is independent of flow depth. For the specific example of the Einstein-Brown equation ($p = 3$), (B7) simplifies to

$$q_s = [F_2 n^{1.8}] q^{1.8} S^{2.1} \quad \text{(B8)}$$

Note that for the wide channel, (B7) and (B8) are exact and require no approximation. This is also the case for overland flow/unit width and for the triangular channel, however, some very small approximations are required to reformulate the sediment transport for the general cross section into (B1).

The geomorphologic time scales of interest in landscape evolution are of the order of thousands of years, certainly much longer than the time scale of individual runoff and erosion events. Temporal averaging of (B1) produces a modified version of this equation, where the discharge is the mean peak discharge from a partial duration flood frequency analysis. A new value of the multiplicative constant β_1 is obtained that is a function of the moments of the distribution of flood events. The long-term average sediment transport rate (per unit width) from averaging over erosion events is

$$\bar{q}_s = \beta_1 \left[\bar{T}_p \lambda \int_{-\infty}^{\infty} [q'(t')]^{m_1}\, dt'\right] \cdot \left[1 + m_1(m_1 - 1)\frac{\sigma_{q_p}^2}{\bar{q}_p^2} + m_1(m_1-1)(m_1-2) \cdot \frac{\gamma_{q_p}(\sigma_{q_p}^2)^{3/2}}{\bar{q}_p^3} + m_1 \frac{\sigma_{q_p T_p}^2}{\bar{q}_p \bar{T}_p}\right] \bar{q}_p^{m_1} S^{n_1} \quad \text{(B9)}$$

where

- t time;
- $q(t)$ discharge during the hydrograph;
- $\bar{T}_p$ mean duration of runoff hydrographs;
- λ rate of occurrence of runoff events;
- $\bar{q}_p$ mean peak discharge per unit width over all the hydrographs that carry significant sediment load from flood frequency analysis;

$q'(t) = q(t)/\bar{q}_p$
$t' = t/\bar{T}_p$

$\sigma^2_{q_p}$ the variance of the peak discharge;
$\sigma^2_{q_p T_p}$ covariance between the peak discharge and the length of the flood hydrograph;
γ_{q_p} skewness coefficient of q_p.

The skewness coefficient is given by

$$\gamma_{q_p} = \frac{\frac{1}{N}\sum_{i=1}^{N}(q_p - \bar{q}_p)_i^3}{(\sigma^2_{q_p})^{3/2}}$$

Acknowledgments. Discussions with D. Tarboton are acknowledged and appreciated. This work results from multiple efforts funded, in parts, by the National Science Foundation (grant 8513556-ECE), the Army Research Office (DAAL03-89-K-0151), a cooperative agreement with the University of Florence through the National Research Council of Italy, and the National Weather Service (Cooperative Research Agreement NA86AA-H-HY123). We also acknowledge the Australian Water Research Advisory Council.

REFERENCES

Abrahams, A. D., Channel networks: A geomorphological perspective, *Water Resour. Res.*, *20*(2), 161–168, 1984.

Acton, F. S., *Numerical Methods that Work*, Harper and Row, New York, 1970.

Ahnert, F., Brief description of a comprehensive three-dimensional process-response model for landform development, *Z. Geomorphol. Suppl.*, *25*, 29–49, 1976.

Ahnert, F. (Ed.), Process-response models of denudation at different spatial scales, in *Geomorphological Models: Theoretical and Empirical Aspects, Catena Suppl.*, *10*, Catena Verlag, Cremlingen-Desdelt, Germany, 1987.

Andrews, D. J., and R. C. Bucknam, Fitting degradation of shoreline scarps by a nonlinear diffusion model, *J. Geophys. Res.*, *92*(B12), 12,857–12,867, 1987.

Begin, Z. B., and S. A. Schumm, Instability of alluvial valley floors: A method for its assessment, *Trans. ASAE*, *22*(2), 347–350, 1979.

Culling, W. E. H., Soil creep and the development of hillslide slopes, *J. Geol.*, *71*(2), 127–161, 1963.

Davis, W., *Geographical Essays*, Dover, New York, 1909.

Dietrich, W. E., C. J. Wilson, and S. L. Reneau, Hollows, colluvium, and landslides in soil-mantled landscapes, in *Hillslope Processes, Binghampton Ser.*, edited by A. D. Abrahams, pp. 361–388, Allen and Unwin, Boston, Mass., 1986.

Dunne, T., Runoff production in a humid area, Ph.D. thesis, Johns Hopkins University, Baltimore, Md., 1969.

Dunne, T., Formation and controls of channel networks, *Prog. Phys. Geogr.*, *4*, 211–239, 1980.

Dunne, T., Hydrology, mechanics and geomorphic implications of erosion by subsurface flow, in *Groundwater Geomorphology*, edited by C. G. Higgins, Geological Society of America, Boulder, Colo., 1989.

Gilbert, G. K., The convexity of hillslopes, *J. Geol.*, *17*, 344–350, 1909.

Havlena, J. A., and G. W. Gross, Climatic and tectonic controls on a semiarid ephemeral drainage system, *Eos Trans. AGU*, *69*(16), 363, 1988.

Henderson, F. M., *Open Channel Flow*, MacMillan, New York, 1966.

Horton, R. E., Erosional development of streams and their drainage basins; Hydrophysical approach to quantitative morphology, *Geol. Soc. Am. Bull.*, *56*, 275–370, 1945.

Huggett, R. J., Dissipative systems: Implications for geomorphology, *Earth Surf. Processes and Landforms*, *13*, 45–49, 1988.

Kirkby, M. J., Hillslope process-response models based on the continuity equation, in *Slopes: Form and Process, Spec. Publ. 3*, pp. 15–30, Institute of British Geographers, London, 1971.

Kirkby, M. J., Modelling some influences on soil erosion, landslides and valley gradient on drainage density and hollow development, in *Geomorphological Models: Theoretical and Empirical Aspects, Catena Suppl. 10*, edited by F. Ahnert, Catena Verlag, Cremlingen-Desdelt, Germany, 1987.

Loewenherz, D. S., Length scale-dependent material transport and the initiation of surface drainage: A modified linear stability theory, *Eos Trans. AGU*, *71*(17), 515–516, 1990.

Meinhardt, H. A., *Models of Biological Pattern Formation*, Academic, San Diego, Calif., 1982.

Montgomery, D. R., and W. E. Dietrich, Where do channels begin?, *Nature*, *336*, 232–234, 1988.

Montgomery, D. R., and W. E. Dietrich, Source areas, drainage density, and channel initiation, *Water Resour. Res.*, *25*(8), 1907–1918, 1989.

Morisawa, M., Development of drainage systems on an upraised lake floor, *Am. J. Sci.*, *262*, 340–354, 1964.

Patton, P. C., and S. A. Schumm, Gully erosion, Northwestern Colorado: A threshold phenomenon, *Geology*, *3*, 88–90, 1975.

Priest, R. F., J. M. Bradford, and R. G. Spomer, Mechanisms for erosion and sediment movement from gullies, in *Proceedings of the Sediment Yield Workshop, USDA Sedimentation Laboratory, Oxford, Mississippi, November 28–30, 1972*, Agricultural Research Service, U.S. Department of Agriculture, Washington, D. C., 1975.

Rodriguez-Iturbe, I., and J. B. Valdes, The geomorphologic structure of hydrologic response, *Water Resour. Res.*, *15*(6), 1409–1420, 1979.

Schumm, S. A., Evolution of drainage systems and slopes in badlands at Perth Amboy, New Jersey, *Geol. Soc. Am. Bull.*, *67*, 597–646, 1956.

Shreve, R. L., Statistical law of stream numbers, *J. Geol.*, *74*, 17–37, 1966.

Smith, T. R., and F. P. Bretherton, Stability and the conservation of mass in drainage basin evolution, *Water Resour. Res.*, *8*(6), 1506–1529, 1972.

Strahler, A. N., Quantitative geomorphology of drainage basins and channel networks, in *Handbook of Applied Hydrology*, edited by V. T. Chow, pp. 4-39 to 4-76, McGraw-Hill, New York, 1964.

Tarboton, D. G., R. L. Bras, and I. Rodriguez-Iturbe, The fractal nature of river networks, *Water Resour. Res.*, *24*(8), 1317–1322, 1988.

Tarboton, D. G., R. L. Bras, and I. Rodriguez-Iturbe, The analysis of river basins and channel networks using digital terrain data, *Tech. Rep. 326*, R. M. Parsons Lab., Dep. of Civ. Eng., Mass. Inst. of Technol., Cambridge, 1989.

Tokunaga, E., Consideration on the composition of drainage networks and their evolution, *Geogr. Rep. 13*, Tokyo Metropolitan University, Japan, 1978.

Vanoni, V. A., *Sedimentation Engineering*, American Society of Civil Engineers, New York, 1975.

Willgoose, G. R., Physically based modelling of landscape evolution and channel network extension: Some supercomputer experiences, Second Australian Supercomputer Conference, Aust. Nucl. Sci. and Technol. Organ., Lucas Heights, Aust., Dec. 13–15, 1989.

Willgoose, G. R., R. L. Bras, and I. Rodriguez-Iturbe, A physically based channel network and catchment evolution model, *TR322*, Ralph M. Parsons Lab. for Water Resour., Dep. of Civ. Eng., Mass. Inst. of Technol., Cambridge, 1989.

Willgoose, G. R., R. L. Bras, and I. Rodriguez-Iturbe, A coupled channel network growth and hillslope evolution model, 2, Nondimensionalization and applications, *Water Resour. Res.*, this issue (*a*).

Willgoose, G. R., R. L. Bras, and I. Rodriguez-Iturbe, A physical explanation of an observed link area-slope relationship, *Water Resour. Res.*, this issue (*b*).

R. L. Bras, Ralph M. Parsons Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139.

I. Rodriguez-Iturbe, Instituto Internacional de Estudios Avanzados, P.O. Box 17606, Parque Central, Caracas, Venezuela.

G. Willgoose, Department of Civil Engineering and Surveying, University of Newcastle, Newcastle, New South Wales 2308, Australia.

(Received June 11, 1990;
revised March 15, 1991;
accepted March 26, 1991.)