The reduced major axis of a bivariate sample

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Summary

In situations such as allometry where a line is to be fitted to a bivariate sample but where an asymmetric choice of one or other variable as regressor cannot be made, the reduced major axis is often used. Existing tests of the slope of this line, particularly between samples, are not sufficiently accurate in view of the scarcity of the material to which such methods are often applied. Alternative test statistics are suggested and some of their properties derived from a computer implementation of k statistics.

Some key words: Allometry; k statistic; Regression.

1. Introduction

One often wishes to describe the relationship between two observed random variables without, in the usual regression terminology, having to specify one as dependent on the other. A typical case, in fact the one which led to this paper, is in allometry where the variables are anatomical measurements, the relationship between which determines shape and may be used as a basis for comparison between species. After suitable transformations, usually logarithmic, have been applied, some measure of the slope of the bivariate scatter plot is required that treats both variables symmetrically. Unless there are sufficient grounds for specifying an underlying model with estimable parameters a possible choice is the line whose sum of squared perpendicular distances from the sample points is a minimum, and it is well known that this is given by the eigenvector corresponding to the larger eigenvalue of the sample dispersion matrix, the smaller eigenvalue in this two variable case being the minimized sum of squares.

For the bivariate normal distribution this line is the major axis of the ellipses of constant probability, and so has come to be called the major axis of the bivariate sample. Although invariant under rotation the major axis is altered in a complicated way by changes of scale and in practice preference in the specialist literature on allometry has been given to the line obtained by normalizing the variables to unit standard deviations, finding the major axis, and transforming back to the original scales of measurement. This has come to be called the reduced major axis. The purpose of this paper is to suggest some more precise methods of testing simple hypotheses about the reduced major axis than have hitherto been available.

2. An estimator of functional relationship

If random variables X and Y have variances $\sigma_x^2$ and $\sigma_y^2$, and correlation $\rho$, then it can easily be shown, supposing for simplicity that $\sigma_x^2 > \sigma_y^2$ and $\rho > 0$, that the major axis has slope $\theta$ relative to the x axis given by

$$\tan \theta = \beta_{maj} = \gamma \pm (\gamma^2 + 1)^{1/2},$$

where $\gamma = \frac{\beta - 1/\beta}{\rho}$ and $\beta = \sigma_y/\sigma_x$. If the variables are standardized, $\beta = \beta_{maj} = 1$, so that after restoring the original scales the reduced major axis has slope $\beta$. With the usual notation
for the two regression lines, $\beta_{yx} = \beta\rho$ and $\beta_{xy} = \beta/\rho$. The reduced major axis is thus the geometric mean of the two regression lines and it is not difficult to show that

$$\beta_{xy} > \beta_{maj} > \beta > \beta_{yx}$$

with equality for $\rho = 1$. If $(x_i, y_i) (i = 1, \ldots, N)$ is a random sample from a population with parameters as above, and if $s_{xy}^2, s_y^2$ and $r$ are the corresponding sample values, then the reduced major axis, $\beta = \sigma_y/\sigma_x$, may be estimated by $b = s_y/s_x$.

Ricker (1973), in an extensive discussion of regression methods in morphometry, advocates use of the reduced major axis as an estimator of an underlying functional relationship. This recommendation was justified earlier by Sprent (1969, p. 38) in discussing the well-known functional relationship problem $x_i = \xi_i + \delta_i, y_i = \eta_i + \epsilon_i$, the $\xi_i$ and $\epsilon_i$ being random variables with zero means, and the $\xi_i$ and $\eta_i$ unobservable parameters satisfying a linear relationship $\eta = \alpha + \beta \xi$, where $\alpha$ and $\beta$ are also unknown. Sprent shows that the sample reduced major axis converges with increasing sample size to a value given by

$$b^2 \to \frac{\beta^2 + \sigma_x^2}{V + \sigma_y^2}$$

as $\Sigma \xi_i^2/n \to V$, a reasonable assumption in practice, because very large values of $\xi_i$ nearly always have zero probability. Thus if there is a wide spread of observed points with comparatively small departures and $\beta$ is not too small the inconsistency of $b$ as an estimator of $\beta$ will not be too large, and the reduced major axis will be a reasonable estimate of the underlying functional relationship.

3. Examples of the test statistics

In the normal case the distribution of $b$ can be obtained quite easily and Finney (1938) has shown how this can be used to test hypotheses such as that $\beta$ is a given constant; his test, however, depends on knowing $\rho$ and is not very sensitive if $\rho$ has to be estimated. Another test is given below. Untransformed, the distribution of $b$ is not symmetric about its mean value, and its variance depends on the mean. Kermack & Haldane (1950) showed that to $O(N^{-1})$

$$\text{var}(s_y/s_x) = \frac{(\sigma_x^2/\sigma_y^2)(1 - \rho^2)}{N - 1},$$

which curiously is the same as $\text{var}(b_{yx})$. Ricker mentions asymmetry of distribution as a difficulty in practical use of the line in morphometry, but the test given here overcomes the problem as it is shown that $\log b$ is distributed symmetrically about its mean with variance $(1 - \rho^2)/n$ to $O(n^{-1})$, where $n = N - 1$, and furthermore that $\log b$ is uncorrelated with $(1 - r^2)$. By analogy with Student's $t$ the obvious test statistic to consider in the one-sample case is thus

$$T = \frac{|\log b - \log \beta|}{\sqrt{(1 - r^2)/n}},$$

whose distribution is asymptotically normal but can be approximated more closely by the method described in §§ 4 and 5.

In practice one does not often need to test the sample line against a known value but rather to compare lines derived from different populations. The appropriate test statistic is

$$T_{12} = \frac{|\log b_1 - \log b_2|}{\sqrt{(1 - r_1^2)/n_1 + (1 - r_2^2)/n_2}},$$

whose distribution is approximated in § 6.
As an illustration of how this is used in practice consider the values $r_1 = 0.8$, $n_1 = 20$, $r_2 = 0.5$, $n_2 = 10$, $b_1 = 0.85$, $b_2 = 0.5$, which are not untypical of those obtained from scarce anatomical material. In the notation of §6 we have that

$$\lambda_1 = (1 - r_1^2)/n_1 = 0.018, \quad \lambda_2 = (1 - r_2^2)/n_2 = 0.075,$$

$$\mu_1 = r_1^2/n_1 = 0.032, \quad \mu_2 = r_2^2/n_2 = 0.025.$$

The test statistic is thus

$$T_{12} = \frac{|\log 0.85 - \log 0.5|}{\sqrt{(\lambda_1 + \lambda_2)}} = 1.74$$

and substitution in formula (6.2) gives $\text{var}(T_{12}) = 1.189$. The distribution of $T_{12}$ is shown to be approximated by the $t$ distribution with degrees of freedom

$$v = 2 + 2/(\text{var}(T_{12}) - 1) = 12.57$$

and the sample value of 1.74 thus lies just below the 0.1 point.

In a paper that has been quite influential in the literature on allometry Imbrie (1956) suggested for testing equality of reduced major axes between two samples the test statistic

$$T_{12} = \frac{|b_1 - b_2|}{\{b_1^2(1 - r_1^2)/N_1 + b_2^2(1 - r_2^2)/N_2\}^{1/2}}$$

to be considered as having approximately a standardized normal distribution. With the sample values used above we obtain a value for Imbrie's test statistic of 2.04, just significant in tables of the normal integral at the 0.05 point, to compare with the corresponding probability for $T_{12}$ of slightly more than 0.1. Clearly for large samples and equal correlations the discrepancy will be less, but in view of the unavoidably small samples from archaeological sites to which these methods are often applied it was considered useful to approximate the distribution of $T_{12}$ to as high a degree as possible. Sampling experiments have shown the tail area approximations derived in §§5 and 6 of this paper to give values in error by less than 0.01 at the 0.1 point even for correlations as large and sample sizes as small as those above.

This problem of finding a good two-sample test for the reduced major axis was drawn to my attention by Dr B. A. Wood of the Middlesex Hospital Medical School, University of London. As part of a study of sexual dimorphism he measured bones and teeth in a number of different primates. Choosing two of his variables, glabella to opisthocranion, i.e. a measure of cranial size, $x$ and maxillary canine base area, $y$, in male and female gorillas we have, after making log transformations of the raw data, Table 1. Using the reduced major axis as a

| Table 1. Summary statistics for Wood's data |
|---|---|
| | Males | Females |
| $N$ | 20 | 17 |
| $s_x$ | 0.0335 | 0.0188 |
| $s_y$ | 0.0549 | 0.0572 |
| $r$ | 0.0158 | 0.0319 |
| $b = s_y/s_x$ | 1.8377 | 3.0388 |
| $\beta_{max}$ | 65.0094 | 9.0845 |

measure of allometric relationship between these two variables do we find any evidence that it is different for males and females? A calculation similar to that above gives $T_{12} = 1.922$ to be compared with the $t$ distribution on 20.3 degrees of freedom; a tail area probability of
0.07 providing little evidence against the null hypothesis. Note that in spite of the scaling down effect of taking logs the major axes are large and substantially different due to the low correlations.

In the following two sections the tests used above are derived. Similar methods could be used to give tests of the slopes of the major axes themselves, but these would be more complicated as they involve the population variances.

4. Method of deriving moments

For the reasons given we consider in the one-sample case the test statistic

\[ T = \frac{\log b - \log \beta}{\sqrt{\frac{(1 - r^2)}{n}}}. \tag{4-1} \]

While it seems difficult to obtain the distribution of this explicitly in a usable form, its moments can be found by expanding in terms of \( k \) statistics and using published tables of their sampling cumulants, a standard procedure for such problems set out by Kendall & Stuart (1977, Ch. 13). For high order terms the algebra is heavy and error-prone and it was decided in this case to write a computer program to carry out the algebraic manipulation. The method is of general application and so details are being submitted for publication separately. A brief outline of the procedure is given below together with a table of moments that will be of use to anyone wishing to examine the properties of statistics other than those considered here, for example the major axis itself. Writing

\[
A = (s_y^2 - \sigma_y^2)/\sigma_y^2, \quad B = (s_x^2 - \sigma_x^2)/\sigma_x^2, \quad C = (r \sigma_x \sigma_y - \rho \sigma_x \sigma_y)/(\rho \sigma_x \sigma_y),
\]

\[
F = \frac{1}{2} \{ \log (1 + A) - \log (1 + B) \}, \quad G = (1 + C)^2 (1 + A)^{-1} (1 + B)^{-1} - 1,
\]

we have that

\[
r^2 = \rho^2 (1 + G), \quad T = F n^4 (1 - \rho^2)^{-4} [1 - \rho^2 G/(1 - \rho^3)]^{-4}.
\]

The method involves three computer programs, one of which expands terms such as \( F^i G^j \) in powers of \( A, B \) and \( C \), another of which tabulates the expectation of \( A^r B^s C^t \) by symbolically expanding the cumulant generating function of the Wishart distribution and a third which combines the results of the first two to give the expectation of \( F^i G^j \).

Writing

\[
F^i G^j = \sum_{\alpha, \beta} L_{\alpha \beta} A^\alpha B^\beta C^t,
\]

preliminary calculation shows that \( r + s + t \leq 6 \) is sufficient to account for all terms down to \( O(n^{-3}) \). Furthermore since \( G \) is symmetric in \( A \) and \( B \), which have the same distribution, and since \( A \) and \( B \) appear similarly but with opposite signs in \( F \), the expectation of \( F^i G^j \) will be zero for odd \( i \).

The results for \( i \) even are given in Table 2, where some terms have not been fully cancelled in order to display their structure more clearly. A check on the accuracy of the program can be made by comparing the results for terms of the form \( G^j \) with published moments of the sample correlation coefficient, as given for example by Cook (1951), where the expectation of \( r^4 \) can be checked against \( E[p^4 (1 + G)^2] \), if we remember that \( N = n + 1 \). Terms of the form \( E[F^i] \) can be partially checked against published results for Fisher's \( z \), putting \( \rho = 0 \).
Table 2 enables the moments of any test statistic based on the bivariate normal that can be expressed as a power series in \( F \) and \( G \) to be easily found.

### Table 2. Moments of the functions \( F \) and \( G \)

<table>
<thead>
<tr>
<th>Function</th>
<th>Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G )</td>
<td>( n^{-1} \rho^{-2}(1-\rho^2)^2 \cdot {(1-2\rho^2) + 2n^{-1}(3-4\rho^2 - 4n^{-2} \rho^2(3-14\rho^2 + 12\rho^4)) } )</td>
</tr>
<tr>
<td>( G^2 )</td>
<td>( n^{-1} \rho^{-4}(1-\rho^2)^2 \cdot {(4\rho^4 + n^{-1}(3-36\rho^2 + 56\rho^4) - 2n^{-2}(3-96\rho^2 + 376\rho^4 - 336\rho^6)) } )</td>
</tr>
<tr>
<td>( G^3 )</td>
<td>( n^{-2} \rho^{-6}(1-\rho^2)^3 \cdot {(12\rho^6(3-8\rho^2) + 3n^{-1}(5-174\rho^2 + 824\rho^4 - 880\rho^6)) } )</td>
</tr>
<tr>
<td>( G^4 )</td>
<td>( n^{-2} \rho^{-8}(1-\rho^2)^4 \cdot {(45\rho^8 + 24n^{-1} \rho^4(15-124\rho^2 + 184\rho^4)) } )</td>
</tr>
<tr>
<td>( G^2 )</td>
<td>( n^{-2} \rho^{-10}(1-\rho^2)^5 \cdot 240\rho^4(5-14\rho^2) )</td>
</tr>
<tr>
<td>( G^3 )</td>
<td>( n^{-3} \rho^{-12}(1-\rho^2)^6 \cdot 980\rho^6 )</td>
</tr>
</tbody>
</table>

| \( F^2 \) | \( n^{-1}(1-\rho^2)^2 \cdot \{(1+n^{-1}(1+\rho^2) + \frac{2n^{-1}(1+\rho^2 + 4\rho^4)) \} \) |
| \( F^4 \) | \( n^{-4}(1-\rho^2)^4 \cdot \{(3 + 2n^{-1}(4 + 5\rho^2)) \} \) |
| \( F^6 \) | \( n^{-3}(1-\rho^2)^6 \cdot 990 \) |
| \( F^3 G \) | \( n^{-2} \rho^{-2}(1-\rho^2)^2 \cdot \{(1-4\rho^2) + n^{-1}(1 + 11\rho^2 - 30\rho^4)) \} \) |
| \( F^4 G^2 \) | \( n^{-2} \rho^{-4}(1-\rho^2)^3 \cdot \{(4\rho^4 + n^{-1}(3-44\rho^2 + 100\rho^4)) \} \) |
| \( F^5 G \) | \( n^{-3} \rho^{-6}(1-\rho^2)^4 \cdot \{(36\rho^6 - 120\rho^8) \} \) |
| \( F^2 G^2 \) | \( n^{-2} \rho^{-8}(1-\rho^2)^5 \cdot 48\rho^8 \) |
| \( F^2 G \) | \( n^{-3} \rho^{-10}(1-\rho^2)^6 \cdot 3(1-6\rho^2) \) |
| \( F^4 G^2 \) | \( n^{-3} \rho^{-12}(1-\rho^2)^8 \cdot 12\rho^8 \) |

### 5. DISTRIBUTION OF THE SINGLE-SAMPLE STATISTIC

Expanding \( T^2 = F^4 n^4(1-\rho^2)^4(1-\rho^2 G/(1-\rho^2))^{-4} \) as a power series in \( F^4 G^2 \), taking expectations and substituting from Table 2, we find for the central moments of \( T \)

\[
\begin{align*}
\mu_2 &= 1 + n^{-1}(2 + \rho^2) + 3n^{-2}(14 + 11\rho^2 + 2\rho^4) + O(n^{-3}), \\
\mu_4 &= 3 + n^{-1}(14 + 10\rho^2) + O(n^{-3}), \\
\mu_6 &= 15 + O(n^{-1})
\end{align*}
\]

which gives \( \gamma_2 = n^{-1}(2 + 4\rho^2) + O(n^{-3}) \).

The similarity to Student's \( t \) suggests that we should assign to \( T \) degrees of freedom \( \nu \) given by

\[
\nu = 2 + n/(1 + \frac{1}{2}\rho^2),
\]

where \( \rho^2 \) if unknown will be replaced by \( r^2 \).

Since \( \gamma_2 \) for the \( t \) distribution is \( 6/(\nu - 4) \), which in this case reduces to \( (6 + 3\rho^2)/(n - 2 - \rho^2) \), it is clear that the approximation is going to be fat in the tails compared with the true distribution of \( T \).

Another approach is to use the Gram–Charlier approximation to the distribution function given in this case by

\[
F(T) = \Phi(T) + n^{-1} T^2 \Phi(T) \{(1 - \frac{1}{2}\rho^2) + (1 + 2\rho^2)(T^2 - 3)/12, \}
\]

where \( \phi \) and \( \Phi \) are the normal probability density and cumulative distribution functions respectively, and again \( \rho^2 \) will usually have to be estimated.

A sampling experiment carried out to check these results showed that there was little to choose between the two approximations for sample sizes of 10 or more. Twenty thousand replications were made for several values of \( \rho \) and \( n \). Both approximations tended to overestimate the tail probability; for large values of \( \rho \) and \( n = 10 \) the error was about 0.006 at the 0.05 point, the Gram–Charlier method being slightly the more accurate.
6. THE TWO-SAMPLE STATISTIC

Where the null hypothesis is equality of reduced major axis between samples, the test statistic is

$$T_{12} = \frac{|\log b_1 - \log b_2|}{\{(1-r_1^2)/n_1 + (1-r_2^2)/n_2\}^{1/2}}$$

(6.1)

whose moments of even order can be found if we expand

$$T_{12}^p = \{F_1 - F_2\}^{2p}(1 - \rho_1^2)/n_1 + (1 - \rho_2^2)/n_2)^{-\lambda}
\left\{1 - \frac{\rho_1^2 G_1/n_1 + \rho_2^2 G_2/n_2}{(1 - \rho_1^2)/n_1 + (1 - \rho_2^2)/n_2}\right\}^{1-\lambda}$$

in powers of $F_1$, $F_2$, $G_1$ and $G_2$. If we substitute from Table 1, we obtain $O(n^{-1})$

$$\text{var}(T_{12}) = 1 + 2(\lambda_1^2 + \lambda_2^2)/\lambda_1 + \lambda_2 + \{3(\lambda_1^2 + \mu_1 + \lambda_2^2 + \mu_2) + \lambda_1 \lambda_2 (\mu_1 + \mu_2)\}/(\lambda_1 + \lambda_2)^2,$$

$$E(T_{12}^4) = 3 + (14\lambda_1^2 + 12\lambda_2^2 + 12\lambda_1 \lambda_2 + 14\lambda_1 \lambda_2 + 24\mu_1 + \lambda_2^2 + 24\mu_2 + \lambda_2^2 + 6\mu_1 \lambda_1 \lambda_2 + 6\mu_2 \lambda_1 \lambda_2)/(\lambda_1 + \lambda_2)^4,$$

where $\lambda_i$ and $\mu_i$ stand for $(1 - \rho_i^2)/n_i$ and $\rho_i^2/n_i$ respectively, $\rho_i$ of course usually being estimated by $r_i$.

As in the single-sample case a Gram–Charlier expansion can be used to approximate the distribution or alternatively, a simpler method to compute by hand, the $t$ distribution can be used with degrees of freedom

$$\nu = 2 + 2/\{\text{var}(T_{12}) - 1\}.$$

Again a sampling experiment was carried out to check the results. At the 0.05 point the tail area was overestimated by 0.014 for sample sizes of 5 and 10 and correlations of 0.3 and 0.8, with progressively better results for larger samples and smaller correlations.

The problem tackled in this paper arose out of the work of Dr B. A. Wood of the Middlesex Hospital Medical School, University of London, on sexual dimorphism in primates. I should like to thank him for raising the problem in the first place and making the data available, and also to acknowledge the help of a referee in pointing the way to further literature on the subject.

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