A RELATIONSHIP BETWEEN CIRCUMFERENCE AND WEIGHT IN TREES AND ITS BEARING ON BRANCHING ANGLES.

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When a tree, at a point where the circumference is $c_0$, divides into two branches ($c_1$ and $c_2$), what relationship exists between $c_0$ and $c_1 + c_2$? In order to answer this question, which has a definite bearing on problems of tree form, it is convenient to investigate first the relationship between the circumference at some point and the weight, $w$, of all the parts of the tree peripheral to this point.

Accordingly measurements were made, 116 in all, on nine kinds of trees; namely, aspen, bitternut, hickory, oak, ash, maple, cedar, hornbeam, and beech. The largest tree measured had a circumference of 56.4 cm. where cut, and the whole tree weighed 120 kg. The smallest measurements were made on the stems of leaves,—for example, circumference of stem = 0.25 cm., weight of leaf = 0.18 gm. All the data thus obtained are included in Fig. 1.

Our procedure was of the simplest character. Whole trees of varying size, or branches, or leaves, were taken entirely at random from the vicinity. The only criterion of selection was that the specimen should not appear to have been recently injured. The circumference was measured by a tape, encircling the bark, at the point of section; or, for small specimens, the diameter was measured by calipers and the circumference subsequently calculated. The specimen was then weighed on one of three balances according to size. The season was midsummer, 1926; the place, Grindstone Island, N. Y.

Plotting logarithms; i.e., log (weight in gm.) vs. log (circumference in cm.), the points fall close to a straight line. A statistical treatment yields the following numerical characteristics:

\[
\begin{align*}
\text{Mean value of } \log c \text{ in the observations} & = 0.161 \\
\text{" " " " } \log w \text{ " " " " } & = 1.250
\end{align*}
\]
RELATION BETWEEN CIRCUMFERENCE AND WEIGHT

Standard deviation of log $c$ = 0.507

" " " log $w$ = 1.263

Mean product of simultaneous deviations = 0.637

Correlation coefficient $r$ = 0.99

From these figures one obtains, for the best linear relation between log $w$ and log $c$, the equation:

$$\log w = 2.49 \log c + 0.850; \text{ or, } w = 7.08 c^{1.49} \quad (1)$$

Fig. 1. The line is drawn according to equation (2). The trend toward a cube law relation among the observations appearing in the lower left portion of the chart may be significant.

The probable error of log $w$ is $\pm 0.08$, i.e., if from the measured circumferences estimates of the weight are calculated by equation (1), then half of the actual observed weights fall within the limits $+20$ per cent and $-17$ per cent of the calculated values. It will be seen from Fig. 1 that the error is greater for the small pieces, and less
for the large pieces. We observed also that deviations from equation (1) occurring in various parts of the same individual tree are sufficiently large to mask any systematic differences between the different kinds of trees that were studied. Furthermore, the equation holds as well for stems bearing nuts as for stems bearing leaves.

Returning to the opening question, the solution, inherent in equation (1), is given by the relation:

\[ c_0^{2.49} = c_1^{2.49} + c_2^{2.49} \]  

(2)*

This follows from the fact that, if a main stem or trunk is cut near a point of branching and weighed \((w_0)\), and then if the branches are weighed separately \((w_1\) and \(w_2\)), \(w_0\) must equal \(w_1 + w_2\).

Equation (2) describes, for the class of trees studied, one special characteristic of branching. The exponent 2.49, being greater than 2, indicates, for example, that the total cross-sectional area of the branches becomes progressively greater at each branching. To express this property we may say that trees follow statistically a "2.5 power law of branching."

Another characteristic of branching is the equation which describes the angles of branching. In a previous paper\(^1\) this problem was discussed in reference to the arterial system in animals, and the following equations, of which two only are independent, were deduced:

\[
\cos x = \frac{c_0^4 + c_1^4 - c_2^4}{2 c_0^2 c_1^2}; \cos y = \frac{c_0^4 - c_1^4 + c_2^4}{2 c_0^2 c_1^2}; \cos (x + y) = \frac{c_0^4 - c_1^4 - c_2^4}{2 c_0^2 c_1^2} 
\]  

(3)

where \(c_0\), \(c_1\), and \(c_2\) are the circumferences of the main stem and the two branches into which it divides; and where \(x\) and \(y\) are the angles made by the branches \((c_1\) and \(c_2)\) with the line of direction of the stem. The angle \((x + y)\) is, of course, the angle included between the two branches. These equations, for our present purpose, may be considered as being deduced from the assumption that the branching system connecting three points shall, for given circumferences of the stem and branches, require the least volume of wood.

*Once obtained, this relation may be roughly checked by simple measurement, without weighing or cutting, on large trees.

\(^1\) The physiological principle of minimum work applied to the angle of branching of arteries, Murray, C. D., *J. Gen. Physiol.*, 1925–26, ix, 835.
If now equations (2) and (3) are combined one can, at the expense of loss of generality, solve directly for the angle as a function of some convenient ratio such as \( c_1/c_2 \) or \( c_1/c_3 \). The steps are shown in the previous paper. But in that paper, instead of a 2.5 power law (equation (2)), a cube law (theoretically deduced for the arterial system) was used. In either case certain qualitative rules hold which describe in words the variations in the angles accompanying variations in the ratio \( c_1/c_2 \), etc. An interesting and convenient illustration of these rules may be seen in the branching of the veins of leaves. There remains only to be observed the fact that, in changing from a cube law to a 2.5 power law, the calculated angles, for any given ratio of circumferences, become smaller,—a fact corresponding to a difference between the branching of arteries and of trees. For example, solving equation (3) for the angle \( (x + y) \) when \( c_1/c_2 = 1 \), we find for a cube law angle \( (x + y) = 75^\circ \), for a 2.5 power law angle \( (x + y) = 59^\circ \). The curve in Fig. 2 shows this relation.
SUMMARY.

Observation reveals a linear relationship between the logarithm of the circumference of a tree, branch, or leaf stem, and the logarithm of the weight of the tree, branch, or leaf. The bearing of this on the angles of branching in trees is discussed.