X6.1

Consider the $n = 2$ state of the hydrogen atom. The object of this problem is to find the eigenvalues of the spin-orbit plus Zeeman Hamiltonian using the $|\,2\ \ell\ m_\ell\ m_s >$ set of basis states (see footnote 20 on page 281 of Griffiths). The usual spin-orbit plus Zeeman perturbation Hamiltonian (**B** is in the z-direction) is (see Equations [6.60] and [6.70])

$$H' = \xi\,\mathbf{L}\cdot\mathbf{S} + \mu_{\mathbf{B}} B(L_z + 2S_z), \quad \xi \equiv \left(\frac{e^2}{8\pi\varepsilon_0}\right)\frac{\hbar^2}{m^2c^2}\left\langle\frac{1}{r^3}\right\rangle$$

Note that we have incorporated $\hbar$ in the definition for ξ so that **L** and **S** become dimensionless. This saves writing a whole lot of "h-bars".

This Hamiltonian perturbs the Bohr plus relativistic correction energy term which we call the "zero" of energy. We will examine how the levels split first when the external magnetic field is increased continuously from to zero to large values.

OK, but how large is "large"? We can change B at will, but we have no control over ξ. It is then reasonable to define ξ as the appropriate energy scaling factor and express all energies in units of ξ, i.e. divide through by ξ. In the literature, ξ is called "the spin-orbit coupling constant". Thus, the Hamiltonian becomes

$$H' = \mathbf{L}\cdot\mathbf{S} + \beta(L_z + 2S_z), \quad \beta \equiv \mu_B B/\xi$$

(a) Show that ξ has units of energy as it should.
(b) List all states in the basis for all possible values of $\ell\ m_\ell\ m_s$. Arrange them in ascending order first by ℓ, then by m_ℓ, then by m_s. Construct a square 8×8 matrix ordered as indicated.
(c) Show that $\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}(L_+S_- + L_-S_+) + L_zS_z$. Use this result to calculate the matrix elements of the spin-orbit interaction. Enter the elements in your matrix.
(d) Add the Zeeman interaction matrix elements.
(e) If you have done everything correctly, you should have a matrix in block diagonal form, 4 1×1 terms and two 2×2 matrices. Find the 8 eigenvalues. Do NOT bother with the eigenvectors.
(f) Plot the eigenvalues as a function of dimensionless parameter β in the region $0 \le \beta \le 3$. You should get a plot that looks like Figure 6.12 with two extra states because Griffiths didn't plot the $\ell = 0$ states. Mark the weak, intermediate and strong field regions. State what criteria you used.

X6.2

Consider the three outer electrons in a free nitrogen atom ($1s^2 2s^2 2p^3$).

(a) How many wavefunctions (microstates) are needed to describe this subspace?
(b) What are the corresponding L, S and J values according to which these states can be grouped? You may have to make educated guesses here.
(c) Verify that the multiplicities add up, i.e. that $\sum (2L+1)\times(2S+1) = \sum (2J+1) =$ answer in part (a).
(d) Find the linear combinations of microstates that are the correct Eigenstates of *L, L_z, S* and S_z.
(e) Construct an energy level diagram showing the splitting of this 3-electron splitting due to spin-orbit coupling. Label all levels with the appropriate terms in spectroscopic notation. Check that the center of energy is conserved.

X8.1

A physics student runs out of gas as he approaches a steep hill. He knows that if he can make it to the top, he can coast down the hill to a gas station. The student calculates that he has enough kinetic energy to go partially up the hill to a level 2 m below the top. Classically, his probability to make it to the gas station is exactly zero. What is his (non-zero) quantum mechanical probability that he can tunnel through the hill and reach the gas station?

Hints: Assume that the relevant part of the hill is a parabola, find its equation and use it in the appropriate equation from the textbook.